ON FULLY COMMUTATIVE ELEMENTS OF TYPE $\tilde{B}$ AND $\tilde{D}$

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Abstract. We define a tower of injections of $\tilde{B}$-type (resp. $\tilde{D}$-type) Coxeter groups $W(\tilde{B}_n)$ (resp. $W(\tilde{D}_n)$). Let $W^c(\tilde{B}_n)$ (resp. $W^c(\tilde{D}_n)$) be the set of fully commutative elements in $W(\tilde{B}_n)$ (resp. $W(\tilde{D}_n)$), we classify the elements of this set by giving a normal form for them. We define a $\tilde{B}$-type tower of Hecke algebras and we use the faithfulness at the Coxeter level to show that this last tower is a tower of injections. We use this normal form to define two injections from $W^c(\tilde{B}_{n-1})$ into $W^c(\tilde{B}_n)$. We then define the tower of affine Temperley-Lieb algebras of type $\tilde{B}$ and use the injections above to prove the faithfulness of this tower. We follow the same track for $\tilde{D}$-type objects.

Affine Coxeter groups; affine Hecke algebra; affine Temperley-Lieb algebra; fully commutative elements.

1. Introduction

Let (W, S) be a Coxeter system. We say that w in W is *fully commutative* if any reduced expression for w can be obtained from any other using only commutation relations among the members of the set S. The present work is part of a series of papers in which we investigate the fully commutative elements in the four infinite series of affine Coxeter groups and the affine Temperley-Lieb algebras of which they index a basis in the following manner:

– we produce a *normal form* for fully commutative elements, by which we mean a reduced expression that has a uniquely prescribed form;
– we rely on this normal form to build injections from the set of fully commutative elements in the rank n group into the same set in rank $n+1$;
– along with this, we bring forth a monomorphism between the rank n group and the rank $n+1$ group in the series, from which we set up a morphism of the corresponding affine Temperley-Lieb algebras;

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– we approximate the latter morphism using the former injections of fully commutative elements and prove eventually its injectivity, thus obtaining a faithful tower of affine Temperley-Lieb algebras.

This study first arose from a question linked to the topological point of view on braid groups: the study – definition, characterization, classification – of Markov traces on affine Temperley-Lieb algebras of type $\tilde{A}$. This was carried out in [1], where the author proves the existence and uniqueness of a Markov trace on a tower of affine Temperley-Lieb algebras of type $\tilde{A}$ [2], relying on the normal form for fully commutative elements of type $\tilde{A}$ that he establishes [3]. Encouraged by a question of Luis Paris, the author went on, in the same spirit, to examine the three other infinite affine families, starting with the production of a normal form for fully commutative elements and the construction of faithful towers of affine Temperley-Lieb algebras: for type $\tilde{C}$ in [4], then for types $\tilde{B}$ and $\tilde{D}$ in the present work.

We keep in our line of sight the topological goal of defining and studying Markov traces on our towers of affine Temperley-Lieb algebras; for instance, in type $\tilde{C}$ such a trace would provide an invariant for knots on a double torus. The study of those towers is a requisite step in such a study. Actually faithfulness is not absolutely necessary, indeed the study of Markov trace for type $\tilde{A}$ was completed before faithfulness was proven [2], but the proofs in [2] are clearly more complicated because faithfulness cannot be used. Hence the present paper as well as [4] are important steps towards the study of new Markov traces. We also remark that the faithfulness of those towers makes an eventual definition of the Markov condition more demanding.

Among common features to the four infinite series of affine Coxeter groups are the infinite dimensionality of the related algebras, affine Hecke algebras and affine Temperley-Lieb algebras, and the lack of parabolicity of one group into the next, say of the rank n group W_n into W_{n+1}. Those two features make it impossible in general to deal with them with the methods used with finite Coxeter groups, so one has to come up with new tools. Heavily used here is the normal form for fully commutative elements that proves powerful. It provides in particular injections from the set of fully commutative elements of a given rank into the next, whereas the existence of a monomorphism from W_n into W_{n+1} is not even obvious. Indeed in the finite case, type A to D, each group W_n is a parabolic subgroup of W_{n+1} so that the natural inclusion from the first into the second restricts to an injective map from fully commutative elements in W_n to fully commutative elements in W_{n+1}. In the infinite case, our injections between sets of fully commutative elements are not restrictions of group morphisms.

Besides this natural use in the area of Temperley-Lieb algebras, enumerating fully commutative elements by a normal form is important in itself (see Stembridge [13] for the three infinite families of finite Coxeter groups), has numerous combinatorics connections (recall that the Catalan number counts fully commutative elements in type A) and has other important applications and consequences. For instance, by providing accessible computations of some Kazhdan-Lusztig polynomials, as was done in [9], it can give a way to attack the study of the 0-1 conjecture, or rather of a fully commutative version of this conjecture since the general formulation has proved wrong [12]. In yet another direction, the author in a forthcoming work will give an explicit Coxeter-length generating function, moreover an *affine-length* generating function (see below), for fully commutative elements, as a direct application of this enumeration (the generating function of the length generating function has been studied in [10] and [5]).

As expanded above, an essential part of this paper is devoted to the construction of two faithful towers in affine types $\tilde{B}$ or $\tilde{D}$: fully commutative elements and affine Temperley-Lieb algebras. Still we also work on two other towers: affine Coxeter groups and affine Hecke algebras. We start by proving the injectivity of the tower of Coxeter groups. This is a crucial step towards injectivity on the Hecke algebra level, which is more technical though.

For affine types $\tilde{A}$ and $\tilde{C}$, we proved earlier the faithfulness of the tower of affine Hecke algebras over the field $\mathbb{Q}[q, q^{-1}]$ ([1, Proposition 4.3.3] and [4, Proposition 3.3]). We used specialization at $q = 1$, that maps Hecke algebras onto group algebras. A similar proof may be used for type $\tilde{D}$; we do not include it because this result is not needed anyway in the rest of the paper. We do however include a new proof of faithfulness for type $\tilde{B}$, which is independent of the ring of definition and goes as well for type $\tilde{C}$, hence extending the result in [4]. It relies on the fact, that may be of independent interest, that in types $\tilde{B}$ and $\tilde{C}$ the morphisms in the towers of Coxeter groups preserve reduced expressions.

Let us pause for this. An element in a Coxeter group is called *rigid* if it has only one reduced expression. In the monomorphisms $W(\tilde{B}_n) \hookrightarrow W(\tilde{B}_{n+1})$ and $W(\tilde{C}_n) \hookrightarrow W(\tilde{C}_{n+1})$ defined below, the Coxeter generators of the first group map to rigid elements of the second; however this does not hold for types $\tilde{A}$ and $\tilde{D}$. We expect our proof for type $\tilde{B}$ to be generic for any reflexion subgroup of a Coxeter group whenever the reflexions are rigid.

Precisely, recall from [7] that a subgroup Y of a Coxeter group (W, S) generated by reflexions is itself a Coxeter group (Y, S_Y) with a canonical set of reflexions S_Y. Paolo

Sentinelli has recently asked whether, in this situation, the corresponding Hecke algebras embed accordingly. Actually it is the existence of such a morphism that is not guaranteed. For instance, there is no morphism of Hecke algebras attached to the monomorphism $\beta : W(\tilde{B}_n) \hookrightarrow W(\tilde{C}_n)$ in §2 (see Remark 2.3). The author conjectures that if there exists a corresponding homomorphism of Hecke algebras, then it is injective. Furthermore, in the case where the elements of S_Y are rigid in (W, S), he expects that it can be proved by elaborating on the proof presented here in type $\tilde{B}$.

The classification of fully commutative elements and the proof of injectivity of the towers of Temperley-Lieb algebras in the four affine types are based on the notion of "affine length": we choose to see any affine Coxeter group as an "affinization" of a finite Coxeter group in the obvious way (but for the case of $\tilde{B}$-type which can be viewed as an extension of either B-type or D-type, we choose the latter for technical reasons). This "affinization" is simply adding a simple reflexion, say a, as follows:

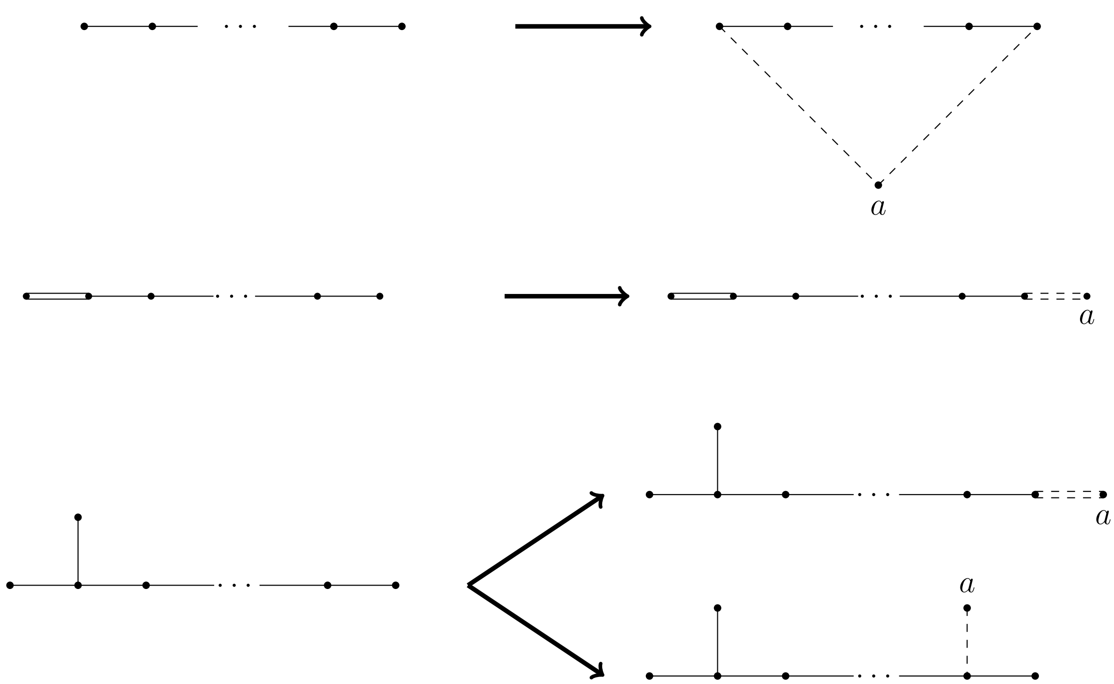

FIGURE 1. A to $\tilde{A}$; B to $\tilde{C}$; D to $\tilde{B}$ and $\tilde{D}$

We thus define the *affine length* of any fully commutative element to be the number of occurrences of a in a (hence any) reduced expression for this fully commutative element. Indeed, let (W, S) be any Coxeter group; for any fully commutative element w in W, let a be any element of S, then the number of occurrences of a in any reduced expression of w is independent of the reduced expression, by Matsumoto's theorem.

Moreover, by distinguishing this a we see simply and directly the construction of towers of each type, based on the morphisms in §2.2 that map, say, a_n to a suitable conjugate of a_{n+1} while fixing the other generators. In types $\tilde{B}$ and $\tilde{C}$, these morphisms E_n and F_n have a remarkable property: the image of a_n is rigid, as mentioned previously. Furthermore, for those two types, the braid relation involving a has even length so that the *affine length* can be defined for any element of the Coxeter group. Eventually we prove that these two morphisms preserve reduced expressions (Corollary 2.7), a key argument for proving the injectivity of the tower of Hecke algebras.

This work is organized as follows. In section 2 we give our notation for Coxeter groups we will be working on, including the "affinization" explained above. We define morphisms between them that give rise to injective towers of affine Coxeter groups of a given type (Corollary 2.2). Furthermore we prove that in types $\tilde{B}$ and $\tilde{C}$ those morphisms preserve reduced expressions (Theorem 2.6 and Corollary 2.7).

In section 3 we define corresponding towers of Hecke algebras. We prove their faithfulness in types $\tilde{B}$ and $\tilde{C}$ over any commutative ring with identity where q is invertible, relying on Corollary 2.7.

In section 4 we recall the normal form given by Stembridge for D-type fully commutative elements [13, Theorem 10.1] and present our normal form for $\tilde{B}$-type and $\tilde{D}$-type fully commutative elements. In both cases we distinguish between two types of elements of affine length at least 2, that we call simply *first type* elements and *second type* elements. Elements of affine length 1 cannot be attached to these two categories without ambiguity so we keep them apart.

We show in section 5 that the normal form of a fully commutative element in $W(\tilde{B}_n)$ (resp.$W(\tilde{D}_n)$) can be transformed into the normal form of a fully commutative element in $W(\tilde{B}_{n+1})$ (resp.$W(\tilde{D}_{n+1})$) in two different ways, that actually coincide for first type elements and elements of affine length one.

We obtain two injective maps I and J that play an essential part in section 6, where we study the morphisms of Temperley-Lieb algebras $Q_n : TL\tilde{B}_n(q) \longrightarrow TL\tilde{B}_{n+1}(q)$ and $P_n : TL\tilde{D}_n(q) \longrightarrow TL\tilde{D}_{n+1}(q)$ induced from the morphisms of Hecke algebras of section 3. Indeed, we can describe the image of a basis element indexed by a fully commutative element w as a linear combination of basis elements in which the terms of highest affine length and highest Coxeter length are indexed by $I(w)$ and $J(w)$ (Lemma 6.2 and Proposition 6.3; Proposition 6.6) or by $I(w)$ and some $I(\bar{w})$ when w has affine length 1 in type $\tilde{D}$ (Lemma 6.7). This implies the faithfulness of the towers of Temperley-Lieb algebras: Theorem 6.4 and Theorem 6.8.

2. Faithful towers of Coxeter groups

Let $(W(\Gamma), S)$ be a Coxeter system with associated Coxeter diagram Γ. Let $w \in W(\Gamma)$ or simply W. We denote by $l(w)$ the length of a (any) reduced expression of w. We define $\mathscr{L}(w)$ to be the set of $s \in S$ such that $l(sw) < l(w)$, in other terms s appears at the left edge of some reduced expression of w. We define $\mathscr{R}(w)$ similarly, on the right.

2.1. **Presentations of D_{n+1}, $\tilde{D}_{n+1}$, $\tilde{B}_{n+1}$, $\tilde{C}_{n+1}$.** For $n \geq 3$ consider the D-type Coxeter group with $n+1$ generators $W(D_{n+1})$, with the following Coxeter diagram:

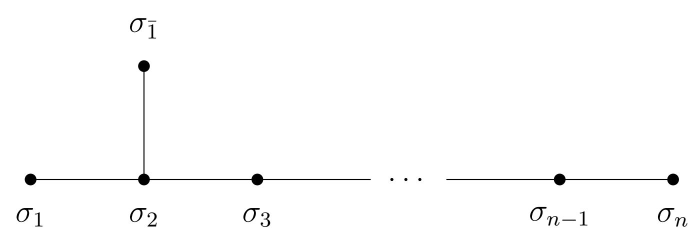

Figure 2. D_{n+1}

Now let $W(\tilde{D}_{n+1})$ be the affine Coxeter group of $\tilde{D}$-type with $n+2$ generators in which $W(D_{n+1})$ could be seen a parabolic subgroup in two ways. We make our choice by presenting $W(\tilde{D}_{n+1})$ with the following Coxeter diagram:

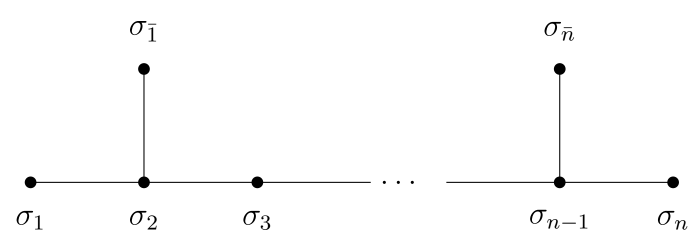

Figure 3. $\tilde{D}_{n+1}$

Let also $W(\tilde{B}_{n+1})$ be the affine Coxeter group of $\tilde{B}$-type with $n+2$ generators in which $W(D_{n+1})$ is naturally a parabolic subgroup, as seen in the following Coxeter diagram:

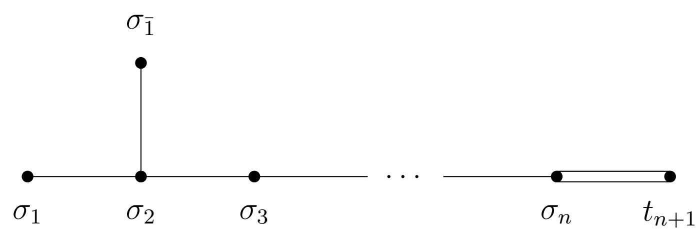

FIGURE 4. $\tilde{B}_{n+1}$

Finally we consider the $\tilde{C}$-type Coxeter group with $n+2$ generators $W(\tilde{C}_{n+1})$, having as Coxeter diagram:

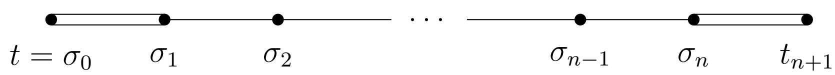

FIGURE 5. $\tilde{C}_{n+1}$

2.2. **Faithfulness of towers of $\tilde{B}$-type and $\tilde{D}$-type affine Coxeter groups.** In [4, Corollary 2.2] the author has defined a faithful tower of $\tilde{C}$-type Coxeter groups by defining the following homomorphism for $n \geq 2$ and proving its injectivity:

$$\begin{aligned} F_n : W(\tilde{C}_n) &\longrightarrow W(\tilde{C}_{n+1}) \\ \sigma_i &\longmapsto \sigma_i \qquad \text{for } 0 \leq i \leq n-1 \\ t_n &\longmapsto \sigma_n t_{n+1} \sigma_n \end{aligned} \tag{1}$$

We define in a similar way the two following homomorphisms:

$$\begin{aligned} E_n : W(\tilde{B}_n) &\longrightarrow W(\tilde{B}_{n+1}) \\ \sigma_i &\longmapsto \sigma_i \qquad \text{for } 1 \leq i \leq n-1 \\ \sigma_{\bar{1}} &\longmapsto \sigma_{\bar{1}} \\ t_n &\longmapsto \sigma_n t_{n+1} \sigma_n \end{aligned} \tag{2}$$

$$\begin{aligned} G_n : W(\tilde{D}_n) &\longrightarrow W(\tilde{D}_{n+1}) \\ \sigma_i &\longmapsto \sigma_i \qquad \text{for } 1 \leq i \leq n-1 \\ \sigma_{\bar{1}} &\longmapsto \sigma_{\bar{1}} \\ \sigma_{\bar{n-1}} &\longmapsto \sigma_n \sigma_{n-1} \sigma_{\bar{n}} \sigma_{n-1} \sigma_n \end{aligned} \tag{3}$$

In [11, §1.13] we see that for $n > 2$, in $W(\tilde{C}_{n+1})$ the subgroup generated by $\{t\sigma_1 t, \sigma_1, \sigma_2, \dots, \sigma_n, t_{n+1}\}$ is exactly $W(\tilde{B}_{n+1})$, and in the latter the subgroup generated by $\{t\sigma_1 t, \sigma_1, \sigma_2, \dots, \sigma_n, t_{n+1}\sigma_n t_{n+1}\}$ is indeed $W(\tilde{D}_{n+1})$. It is convenient to denote respectively β and δ_{n+1} those inclusion maps (that is, $\beta(\sigma_{\bar{1}}) = t\sigma_1 t$ and $\delta_{n+1}(\sigma_{\bar{n}}) = t_{n+1}\sigma_n t_{n+1}$, in such a way that we have the following lemma:

Lemma 2.1. *The diagram commutes for any $n > 2$*

$$\begin{tikzcd}
W(\tilde{C}_n) \arrow[r, hook, "F_n"] & W(\tilde{C}_{n+1}) \\
W(\tilde{B}_n) \arrow[u, hook, "\beta"] \arrow[r, "E_n"'] & W(\tilde{B}_{n+1}) \arrow[u, hook, "\beta"'] \\
W(\tilde{D}_n) \arrow[u, hook, "\delta_n"] \arrow[r, "G_n"'] & W(\tilde{D}_{n+1}) \arrow[u, hook, "\delta_{n+1}"']
\end{tikzcd}$$

Corollary 2.2. *E_n and G_n are injections.*

Remark 2.3. The horizontal morphisms in the previous lemma, namely F_n, E_n and G_n, can be defined on the corresponding braid groups (by mapping t_n to $\sigma_n t_{n+1} {\sigma_n}^{-1}$ for F_n and E_n, or mapping $\sigma_{\overline{n-1}}$ to $\sigma_n\sigma_{n-1}\sigma_{\bar{n}}\sigma_{n-1}^{-1}\sigma_n^{-1}$ for G_n). They thus define morphisms of algebra on the braid group algebras and, after taking quotients, on the Hecke algebras studied in the next section. This is not the case for δ_n or β.

2.3. **Further properties of the towers in type $\tilde{B}$ and $\tilde{C}$.** We now remark that, in the groups $W(\tilde{B}_{n+1})$ and $W(\tilde{C}_{n+1})$, the only braid relation involving t_{n+1} (apart from commutation relations) is

$$t_{n+1}\sigma_n t_{n+1}\sigma_n = \sigma_n t_{n+1}\sigma_n t_{n+1} \tag{4}$$

where the number of occurrences of t_{n+1} is the same on both sides. It follows (recall [6, §1.5 Proposition 5]) that the number of times t_{n+1} occurs in a reduced expression of an element of $W(\tilde{B}_{n+1})$ or $W(\tilde{C}_{n+1})$ does not depend of this reduced expression.

Definition 2.4. *Let $u \in W(\tilde{B}_{n+1})$ or $W(\tilde{C}_{n+1})$. We define the* affine length *of u to be the number of times t_{n+1} occurs in a (any) reduced expression of u. We denote it by $L(u)$.*

In the group $W(\tilde{D}_{n+1})$, the braid relation $\sigma_{n-1}\sigma_{\bar{n}}\sigma_{n-1} = \sigma_{\bar{n}}\sigma_{n-1}\sigma_{\bar{n}}$ prevents us from giving the same definition, except for fully commutative elements as we will see below (Definition 4.5).

We will now examine more closely the monomorphisms $F_n : W(\tilde{C}_n) \longrightarrow W(\tilde{C}_{n+1})$ and $E_n : W(\tilde{B}_n) \longrightarrow W(\tilde{B}_{n+1})$ and show that they preserve the affine length and transform the Coxeter length of some w into $l(w) + 2L(w)$. In other words, when substituting $\sigma_n t_{n+1} \sigma_n$ to t_n in a reduced expression for w, we obtain a reduced expression for the image of w.

We let in this subsection W_n be $W(\tilde{C}_n)$ or $W(\tilde{B}_n)$, with set of generators $S_n = \Sigma \cup \{t_n\}$ while W_{n+1} has $S_{n+1} = \Sigma \cup \{\sigma_n, t_{n+1}\}$ as set of generators. We let I be either F_n or E_n, i.e. I is the monomorphism from W_n to W_{n+1} that is the identity on generators in Σ and maps t_n to $\sigma_n t_{n+1} \sigma_n$. We also let W_Σ be the subgroup of W_n generated by Σ.

Lemma 2.5. *Let x be in $I(W_n)$ then:*

(a) $t_{n+1} x = x t_{n+1}$ *;*

(b) If $\sigma_n x = x \sigma_n$ *then* $I(t_n) x = x I(t_n)$*;*

(c) If $x t_{n+1} \sigma_n = \sigma_n t_{n+1} x$ *then* $I(t_n) x = x I(t_n)$*;*

Proof. (a) Indeed t_{n+1} commutes with generators in Σ and the braid relation (4) expresses commutation with $T(t_n)$. Then Point (a) holds and Point (b) follows immediately.

Using (a) the assumption in (c) can be written $x\sigma_n x^{-1} = t_{n+1}\sigma_n t_{n+1}$, hence by (4):

$$x\sigma_n x^{-1} = \sigma_n t_{n+1} \sigma_n t_{n+1} \sigma_n = (\sigma_n t_{n+1} \sigma_n)\sigma_n(\sigma_n t_{n+1} \sigma_n) = I(t_n)\sigma_n I(t_n).$$

So $I(t_n)x$ commutes with σ_n, hence with $I(t_n)$ by (b), which gives
$I(t_n)I(t_n)x = I(t_n)xI(t_n)$, that is $I(t_n)x = xI(t_n)$. □

We now remember [6, Ch. IV, §1.4]. Given a Coxeter system (W, S), we attach to any finite sequence $\mathbf{s} = (s_1, \cdots, s_r)$ of elements in S, the multiset $\Phi(\mathbf{s}) = \{\{h_1, \cdots, h_r\}\}$ of elements in W defined by:

$$h_j = (s_1 \cdots s_{j-1})\ s_j\ (s_1 \cdots s_{j-1})^{-1} \qquad \text{for } 1 \leq j \leq r.$$

We write $\mathrm{MCard}\,\Phi(\mathbf{s})$ for the multi-cardinal of $\Phi(\mathbf{s})$, i.e.the number of elements in the multiset $\Phi(\mathbf{s})$, namely r here, and $\mathrm{Card}\,\Phi(\mathbf{s})$ for the cardinality of the set underlying $\Phi(\mathbf{s})$. We know from *loc.cit.* Lemma 2 that:

the product $s_1 \cdots s_r$ is a reduced expression of the element $s_1 \cdots s_r$ in W if and only if $\mathrm{MCard}\,\Phi(\mathbf{s}) = \mathrm{Card}\,\Phi(\mathbf{s})$*, that is, all elements in the multiset are distinct.*

We now let $w \in W_n$ and fix a reduced decomposition of w as follows :

$$w = u_0 t_n u_1 \dots u_{s-1} t_n u_s$$

where $u_0, \cdots, u_s$ are fixed reduced decompositions of elements in W_Σ. In particular the affine length of w is s. The image $I(w)$ in W_{n+1} is

$$(*) \qquad I(w) = u_0 \sigma_n t_{n+1} \sigma_n u_1 \dots u_{s-1} \sigma_n t_{n+1} \sigma_n u_s.$$

We use this fixed expression of $I(w)$ to study what we call, by abuse of language, $\Phi(I(w))$. We can describe this multiset as a disjoint union of multisets

$$\Phi(I(w)) = I(\Phi(w)) \bigsqcup T_1 \bigsqcup T_2$$

as follows. We develop the expressions of $u_0, \dots, u_s$ so that $(*)$ reads $I(w) = s_1 \cdots s_r$, then we have three possibilities for an element $h_i = (s_1 \cdots s_{i-1})\ s_i\ (s_1 \cdots s_{i-1})^{-1}$ of $\Phi(I(w))$:

(1) $s_i \in \Sigma$ or $s_i = t_{n+1}$; then $h_i = xXx^{-1}$ where x and X are in $I(W_n)$. Those elements make up exactly the multiset $I(\Phi(w))$.
(2) $s_i = \sigma_n$ and $s_{i+1} = t_{n+1}$; then $h_i = x\sigma_n x^{-1}$ with x in $I(W_n)$. Those elements make up the multiset T_1.
(3) $s_i = \sigma_n$ and $s_{i-1} = t_{n+1}$; then $h_i = x\sigma_n t_{n+1}\sigma_n t_{n+1}\sigma_n x^{-1} = x t_{n+1}\sigma_n t_{n+1} x^{-1}$ with x in $I(W_n)$. Those elements make up the multiset T_2.

We have $\mathrm{MCard}(T_1) = \mathrm{MCard}(T_2) = L(w)$ and $\mathrm{MCard}(I(\Phi(w))) = l(w)$, since w is given in a reduced expression (we will anyway see it in (1) of the following theorem). Hence the multi-cardinal of $\Phi(I(w))$ is $l(w) + 2L(w)$. If the elements of $\Phi(I(w))$ are all distinct, then it is the cardinal.

Theorem 2.6. *Let $w \in W_n$ and take the above conventions. Then*

$$MCard(\Phi(I(w))) = Card(\Phi(I(w))) = l(w) + 2L(w).$$

In particular, substituting $\sigma_n t_{n+1} \sigma_n$ to t_n in a reduced expression for w produces a reduced expression for the image $I(w)$ of w.

Proof. Take h_i and h_j in $\Phi(I(w))$ with $i \neq j$ and suppose that $h_i = h_j$. We have six cases.

(1) h_i and h_j are in $I(\Phi(w))$: contradicts that w is given in a reduced expression and I is injective.

(2) h_i and h_j are in T_1, gives (b) in the above lemma, thus contradicts that w is given in a reduced expression.

(3) h_i and h_j are in T_2, gives (b) in the above lemma, thus contradicts that w is given in a reduced expression.

(4) h_i is in $I(\Phi(w))$ and h_j is in T_1, gives $\sigma_n \in I(W_n)$, hence commutes with t_{n+1} by (a) in the above lemma, contradiction.

(5) h_i is in $I(\Phi(w))$ and h_j is in T_2, gives $t_{n+1}\sigma_n t_{n+1} \in I(W_n)$, hence commutes with t_{n+1} by (a) in the above lemma, contradiction.

(6) h_i is in T_1 and h_j is in T_2, gives (c) in the above lemma, thus contradicts that w is given in a reduced expression. □

Corollary 2.7. *The monomorphisms*

$$F_n : W(\tilde{C}_n) \longrightarrow W(\tilde{C}_{n+1})$$
$$\textit{and} \quad E_n : W(\tilde{B}_n) \longrightarrow W(\tilde{B}_{n+1})$$

satisfy, for any $v \in W(\tilde{C}_n)$ and for any $w \in W(\tilde{B}_n)$:

$$l(F_n(v)) = l(v) + 2L(v) \quad \textit{and } L(F_n(v)) = L(v);$$
$$l(E_n(w)) = l(w) + 2L(w) \quad \textit{and } L(E_n(w)) = L(w).$$

3. The towers of Hecke algebras

Let for the moment K be an arbitrary commutative ring with identity. In what follows, by algebra we mean K-algebra. We recall [6, Ch. IV §2 Ex. 23] that for a given Coxeter graph Γ and a corresponding Coxeter system $(W(\Gamma), S)$, there is a unique algebra structure on the free K-module with basis $\{g_w \mid w \in W(\Gamma)\}$ satisfying, for $s \in S$ and a given $q \in K$:

$$\begin{aligned} g_s g_w &= g_{sw} && \text{if } s \notin \mathscr{L}(w), \\ g_s g_w &= q g_{sw} + (q-1) g_w && \text{if } s \in \mathscr{L}(w). \end{aligned}$$

We denote this algebra by $H\Gamma(q)$ and call it the the Γ-type Hecke algebra. This algebra has a presentation (*loc.cit.*) given by generators $\{g_s \mid s \in S\}$ and relations

$$\begin{aligned} g_s^2 &= q + (q-1) g_s && \text{for } s \in S, \\ (g_s g_t)^r &= (g_t g_s)^r && \text{for } s, t \in S \text{ such that } st \text{ has order } 2r, \\ (g_s g_t)^r g_s &= (g_t g_s)^r g_t && \text{for } s, t \in S \text{ such that } st \text{ has order } 2r+1. \end{aligned}$$

We assume in what follows that q is invertible in K. In this case the first defining relation above implies that g_s, for $s \in S$, is invertible with inverse

$$g_s^{-1} = \frac{1}{q}\, g_s + \frac{q-1}{q}. \tag{5}$$

We consider the Hecke algebras $H\tilde{C}_n(q)$, of type $\tilde{C}_n$, $H\tilde{B}_n(q)$, of type $\tilde{B}_n$, and $H\tilde{D}_n(q)$, of type $\tilde{D}_n$, which correspond respectively to the Coxeter groups $W(\tilde{C}_n)$, $W(\tilde{B}_n)$ and $W(\tilde{D}_n)$. The morphisms F_n, E_n and G_n of §2 have a counterpart in the setting of Hecke algebras, as follows from Remark 2.3 or as can be checked directly, namely the following morphisms of algebras (where we write carefully e_w for the basis elements of the algebras in rank n, to be reminded of the possible lack of injectivity):

$$\begin{aligned} R_n : H\tilde{C}_n(q) &\longrightarrow H\tilde{C}_{n+1}(q) \\ e_{\sigma_i} &\longmapsto g_{\sigma_i} \qquad\qquad \text{for } 0 \le i \le n-1 \\ e_{t_n} &\longmapsto g_{\sigma_n} g_{t_{n+1}} g_{\sigma_n}^{-1}. \end{aligned} \tag{6}$$

$$\begin{aligned} Q_n : H\tilde{B}_n(q) &\longrightarrow H\tilde{B}_{n+1}(q) \\ e_{\sigma} &\longmapsto g_{\sigma} \qquad\qquad \text{for } \sigma \in \{\sigma_1, \sigma_{\bar{1}}, \dots, \sigma_n\} \\ e_{t_n} &\longmapsto g_{\sigma_n} g_{t_{n+1}} g_{\sigma_n}^{-1}. \end{aligned} \tag{7}$$

$$\begin{aligned} P_n : H\tilde{D}_n(q) &\longrightarrow H\tilde{D}_{n+1}(q) \\ e_{\sigma} &\longmapsto g_{\sigma} \qquad\qquad \text{for } \sigma \in \{\sigma_1, \sigma_{\bar{1}}, \dots, \sigma_{n-1}\} \\ e_{\sigma_{\overline{n-1}}} &\longmapsto g_{\sigma_n} g_{\sigma_{n-1}} g_{\sigma_{\bar{n}}} g_{\sigma_{n-1}}^{-1} g_{\sigma_n}^{-1}. \end{aligned} \tag{8}$$

When K is the ring $\mathbf{Q}[q, q^{-1}]$ of Laurent polynomials in q with rational coefficients, we proved in [4, Proposition 3.3] that the morphism R_n is injective, hence producing a faithful tower of Heche algebras of type $\tilde{C}$. Actually for types $\tilde{C}$ and $\tilde{B}$ we prove below injectivity for any ground ring K. This proof cannot be used for type $\tilde{D}$ because the monomorphism G_n defined in Corollary 2.2 does not transform the length in the linear way F_n and E_n do: we have to restrict to $K = \mathbf{Q}[q, q^{-1}]$ and imitate the proofs of [1, Proposition 4.3.3] and [4, Proposition 3.3]. We don't include this since in any case, injectivity at the level of Hecke algebras is not needed to prove injectivity for towers of Temperley-Lieb algebras in §6.

3.1. **Faithfulness of towers of $\tilde{B}$-type and $\tilde{C}$-type Hecke algebras.** For types $\tilde{B}$ and $\tilde{C}$ the key is the following Proposition:

Proposition 3.1. *Let w be any element in $W(\tilde{C}_n)$. Then:*

$$R_n(e_w) = A_w g_{F_n(w)} + \sum_{\substack{x \in W(\tilde{C}_{n+1}),\\ l(x)<l(F_n(w))\\ L(x)\leq L(w)}} \lambda_x g_x,$$

where A_w belongs to $q^{\mathbb{Z}}$ and the λ_x belong to K.

The same holds for $w \in W(\tilde{B}_n)$, replacing R_n by Q_n and F_n by E_n.

Proof. This is a direct consequence of Corollary 2.7. To obtain it we use definitions (6) of R_n and (7) of Q_n and develop the images $R_n(e_w)$ and $Q_n(e_w)$ with the expression (5) for the inverse of g_{σ_n}. □

Theorem 3.2. *Let K be a ring and q be invertible in K. The towers of affine Hecke algebras:*

$$H\tilde{C}_1(q) \xrightarrow{R_1} H\tilde{C}_2(q) \xrightarrow{R_2} \cdots H\tilde{C}_n(q) \xrightarrow{R_n} H\tilde{C}_{n+1}(q) \longrightarrow \cdots$$

$$H\tilde{B}_2(q) \xrightarrow{Q_2} H\tilde{B}_3(q) \xrightarrow{Q_3} \cdots H\tilde{B}_n(q) \xrightarrow{Q_n} H\tilde{B}_{n+1}(q) \longrightarrow \cdots$$

are towers of faithful arrows.

Proof. We write the proof in case $\tilde{C}$, case $\tilde{B}$ is identical. Assuming a non trivial dependence relation

$$(*) \qquad \sum_w \lambda_w R_n(e_w) = 0,$$

we let $m = \max\{L(w) \mid w \in W^c(\tilde{C}_n) \text{ and } \lambda_w \neq 0\}$.

In the development of $(*)$ on the basis $(g_x)_{x\in W(\tilde{C}_{n+1}}$, the terms g_x with non-zero coefficient have $L(x) \leq m$ and, among those, the ones with $L(x) = m$ come exclusively from $\sum_{L(w)=m} \lambda_w R_n(e_w)$. So among those again, using Proposition 3.1, the ones with maximal Coxeter length are the $\lambda_w A_w g_{F_n(w)}$ where $L(w) = m$ and $l(w) = a$, with $a = \max\{l(w) \mid L(w) = m \text{ and } \lambda_w \neq 0\}$. But they are linearly independent, a contradiction. □

4. Full commutativity and normal forms

4.1. **Fully commutative elements.** In a given Coxeter group (W, S), we know that, from a given reduced expression of an element w, we can arrive to any other reduced expression of w only by applying braid relations [6, §1.5 Proposition 5]. Among these relations there are commutation relations: those that correspond to generators t and s in S such that st has order 2.

Definition 4.1. *Let Γ be a Coxeter diagram. Elements of $W(\Gamma)$ for which one can pass from any reduced expression to any other one only by applying commutation relations are called* fully commutative elements*. We denote by $W^c(\Gamma)$, or simply W^c, the set of fully commutative elements in $W = W(\Gamma)$.*

We now consider $W(D_{n+1})$ generated by $\{\sigma_1, \sigma_{\bar{1}}, \dots, \sigma_n\}$. The set $W^c(D_{n+1})$ is described by Stembridge in [13]. We use the notation there and the same assumption that the subword $\sigma_{\bar{1}}\sigma_1$ does not appear (we see it as $\sigma_1\sigma_{\bar{1}}$ for the sake of unicity). For integers $j \geq i \geq 2$ and $k \geq 1$ let:

$$\langle i, j] = \sigma_i\sigma_{i+1}\dots\sigma_j \ ; \ \langle -i, j] = \sigma_i\sigma_{i-1}\dots\sigma_2\sigma_1\sigma_{\bar{1}}\sigma_2\dots\sigma_{j-1}\sigma_j,$$
$$\langle 1, k] = \sigma_1\sigma_2\dots\sigma_k \ ; \ \langle -1, k] = \sigma_{\bar{1}}\sigma_2\dots\sigma_k \ \ ; \ \langle 0, k] = \sigma_1\sigma_{\bar{1}}\sigma_2\dots\sigma_k;$$

so that $\langle -1, 1] = \sigma_{\bar{1}}$ and $\langle 0, 1] = \sigma_1\sigma_{\bar{1}}$. We also let, for convenience and later use: $\langle n+1, n] = 1$. Then every element of $W(D_{n+1})$ has a unique reduced expression of the form $\langle m_1, n_1]\langle m_2, n_2]\dots\langle m_r, n_r]$ with $n \geq n_1 > n_2 > \dots n_r \geq 1$ and $|m_i| \leq n_i$ for $1 \leq i \leq r$ (*loc.cit.* p.1310). Now:

Theorem 4.2. [13, Theorem 10.1] *$W^c(D_{n+1})$ is the set of elements with a reduced expression of the form*

$$\langle m_1, n_1]\langle m_2, n_2]\dots\langle m_r, n_r] \tag{9}$$

with $n \geq n_1 > n_2 > \dots n_r \geq 1$ and $|m_i| \leq n_i$ for $1 \leq i \leq r$, where the occurrences of 1 and -1 alternate and either

(1) $m_1 > \dots > m_s > |m_{s+1}| = \dots = |m_r| = 1$ for some $s \leq r$, or
(2) $m_1 > \dots > m_{r-1} > -m_r \geq 0$, $m_{r-1} > 1$, and $m_r \neq -1$.

We remark that if $r > 1$, then $\langle m_2, n_2]\dots\langle m_r, n_r]$ in (9) above belongs to $W^c(D_n)$. From now on we will mean by *"the form (9)"* the form of a fully commutative element unless we mention otherwise. We notice that if σ_n appears in form (9) above, then it appears necessarily in the first term $\langle m_1, n_1]$ and either it appears only once and we have $n = n_1 \neq -m_1$, or it appears exactly twice and we have $n = n_1 = -m_1$ and

$r = 1$. Now σ_{n-1} can only appear in the two first terms $\langle m_1, n_1]\langle m_2, n_2]$ and this happens if and only if $n_1 \geq n-1$. If σ_{n-1} appears more than once, then in form (9) either $n_2 = n-1$ and $n_1 = n$, in which case the constraints of Theorem 4.2 show that σ_{n-1} appears at most twice (for if $m_2 = -(n-1)$ then $m_1 = n$); or $n_2 < n-1 \leq n_1$ and $m_1 \leq -(n-1)$ so $r = 1$.

Definition 4.3. *An element u in $W^c(D_{n+1})$ is called* $\tilde{B}$-extremal *if σ_n appears in a (any) reduced expression for u. In this case u can be written in form (9) with $n_1 = n$ and either $u = \langle -n, n]$, or $m_1 > -n$ and σ_n appears only once.*

Definition 4.4. *An element u in $W^c(D_{n+1})$ is called* $\tilde{D}$-extremal *if σ_{n-1} appears twice in a (any) reduced expression for u. In this case u can be written in exactly one of the four following forms:*

- $\langle -(n-1), n-1]$ *;*
- $\langle -(n-1), n]$ *;*
- $\langle n, n]\langle -(n-1), n-1]$ *;*
- $\langle m_1, n]\langle m_2, n-1] \ldots \langle m_r, n_r]$ *with $r \geq 2$ and $m_1 \leq n-1$ in addition to the conditions of Theorem 4.2.*

We defined previously the affine length of an element of $W(\tilde{B}_{n+1})$ (Definition 2.4), which we couldn't do on the full group $W(\tilde{D}_{n+1})$ on account of the braid relation

$$\sigma_{n-1}\sigma_{\bar{n}}\sigma_{n-1} = \sigma_{\bar{n}}\sigma_{n-1}\sigma_{\bar{n}}.$$

Since the words here cannot be subwords of any reduced expression of an element of $W^c(\tilde{D}_{n+1})$, it follows (recall [6, §1.5 Proposition 5]) that the number of times $\sigma_{\bar{n}}$ occurs in a reduced expression of an element of $W^c(\tilde{D}_{n+1})$ does not depend of this reduced expression.

Definition 4.5. *Let $u \in W^c(\tilde{D}_{n+1})$. We define the* affine length *of u to be the number of times $\sigma_{\bar{n}}$ occurs in a (any) reduced expression of u. We denote it by $L(u)$.*

4.2. Reduced expressions of elements of $W^c(\tilde{B}_{n+1})$.

Lemma 4.6. *Let w be a fully commutative element in $W(\tilde{B}_{n+1})$ with $L(w) = m \geq 2$ for $n \geq 3$. Fix a reduced expression of w as follows:*

$$w = u_1 t_{n+1} u_2 t_{n+1} \ldots u_m t_{n+1} u_{m+1}$$

with u_i, for $1 \leq i \leq m+1$, a reduced expression of an element in $W^c(D_{n+1})$. Then $u_2, \ldots, u_m$ are $\tilde{B}$-extremal elements and there is a reduced expression of w of the form:

$$w = \langle i_1, n]t_{n+1}\langle i_2, n]t_{n+1} \ldots \langle i_m, n]t_{n+1}v_{m+1} \tag{10}$$

where $v_{m+1} \in W^c(D_{n+1})$ and one of the following holds:

(1) $n+1 \geq i_1 > \cdots > i_s > |i_{s+1}| = \cdots = |i_m| = 1$ *for some* $s \leq m$*, and the occurrences of* 1 *and* -1 *alternate, or*
(2) $n+1 \geq i_1 > \cdots > i_{m-1} > -i_m \geq 0$*,* $i_{m-1} > 1$*, and* $i_m \neq -1$*, or*
(3) $n+1 \geq i_1 \geq i_2 = \cdots = i_m = -n$ *.*

Furthermore, in case (2) we have $v_{m+1} = 1$*.*

Proof. Since t_{n+1} commutes with $W(D_n)$, the fact that the expression is reduced forces u_i to be $\tilde{B}$-extremal for $2 \leq i \leq m$. We use form (9) for u_1 and write it as $u_1 = \langle i_1, n] x_1$, a reduced expression with x_1 in $W^c(D_n)$ and $-n \leq i_1 \leq n+1$. Here x_1 commutes with t_{n+1} hence we get a reduced expression

$$w = \langle i_1, n] t_{n+1} x_1 u_2 t_{n+1} \ldots u_m t_{n+1} u_{m+1}.$$

Again $x_1 u_2 \in W^c(B_{n+1})$ has a reduced expression $\langle i_2, n] x_2$ with $-n \leq i_2 \leq n$ (since $x_1 u_2$ is $\tilde{B}$-extremal) and x_2 in $W^c(D_n)$, and this x_2 commutes with t_{n+1} and can be pushed to the right, leading to

$$w = \langle i_1, n] t_{n+1} \langle i_2, n] t_{n+1} x_2 u_3 t_{n+1} \ldots u_m t_{n+1} u_{m+1}.$$

Proceeding from left to right we obtain formally form (10).

We now fix j, $1 \leq j < m$.

- If $i_{j+1} = n$ we must have $i_j = n+1$ (and $j = 1$) since, with $i_j \leq n$, our expression would contain the braid $\sigma_n t_{n+1} \sigma_n t_{n+1}$, contradicting the full commutativity;
- while if $i_{j+1} = -n$, then if $j = 1$, any i_1 is possible, but if $j > 1$ we must have $i_{j+1} = i_j = -n$ to avoid the braid $t_{n+1} \sigma_n t_{n+1} \sigma_n$.
- If $1 < |i_{j+1}| < n$ the reduced expression of u contains the subexpression $\langle i_j, n] t_{n+1} \sigma_{|i_{j+1}|} = \langle i_j, n] \sigma_{|i_{j+1}|} t_{n+1} = \ldots$ where $\sigma_{|i_{j+1}|}$ can be pushed to the left until we reach, if $i_j \leq |i_{j+1}|$ the braid $\sigma_{|i_{j+1}|} \sigma_{|i_{j+1}|+1} \sigma_{|i_{j+1}|}$, again a contradiction to the full commutativity; hence we have $i_j > |i_{j+1}|$.
- If $|i_{j+1}| = 1$ the same argument, pushing σ_1 or $\sigma_{\bar{1}}$ to the left, past t_{n+1} and further, shows that we must have $i_j > 1$ or $i_j = -i_{j+1}$, otherwise we would get the braid $\sigma_1 \sigma_2 \sigma_1$ or $\sigma_{\bar{1}} \sigma_2 \sigma_{\bar{1}}$.
- Finally if $i_{j+1} = 0$, we must have $i_j > 1$ to avoid the braids $\sigma_1 \sigma_2 \sigma_1$ and $\sigma_{\bar{1}} \sigma_2 \sigma_{\bar{1}}$.

Putting together the above rules gives the possibilities announced and a similar argument provides the last assertion. $\square$

This lemma leads directly to the classification of fully commutative elements in $W(\tilde{B}_{n+1})$.

Theorem 4.7. *Let $n \geq 3$. Let $w \in W^c(\tilde{B}_{n+1})$ with $L(w) \geq 2$. Then w can be written in a unique way as a reduced word of one and only one of the following two forms, for non negative integers p and k:*

First type:

$$w = \langle -i, n] t_{n+1} \left(\langle -n, n]\; t_{n+1}\right)^k \;\; \langle f, n]^{-1} \tag{11}$$

with $-n \leq i \leq n+1$ and $-n \leq f \leq n+1$.

We call such elements <u>quasi-rigid</u> because their reduced expression is unique up to replacing $\sigma_1 \sigma_{\bar{1}}$ by $\sigma_{\bar{1}} \sigma_1$.

Second type:

(12)

$$w = \langle i_1, n]\; t_{n+1} \dots \langle i_p, n]\; t_{n+1} \langle j_1, n]\; t_{n+1} \langle j_2, n]\; t_{n+1} \dots \langle j_k, n]\; t_{n+1}\; w_r \quad \text{if } p > 0,$$

$$w = \langle j_1, n]\; t_{n+1} \langle j_2, n]\; t_{n+1} \dots \langle j_k, n]\; t_{n+1}\; w_r \quad \text{if } p = 0,$$

$$w = \langle i_1, n]\; t_{n+1} \langle i_2, n]\; t_{n+1} \dots \langle i_p, n]\; t_{n+1}\; w_r \quad \text{if } k = 0,$$

with $w_r \in W^c(D_{n+1})$ and

- *if $k > 0$:*
  - *for $1 \leq s \leq k$, we have $j_s = (-1)^{a+s}$ for some integer a, that is to say $j_s = \pm 1$ and 1 and -1 alternate ;*
  - *and $w_r = \langle (-1)^{a+k+1}, r_1] \langle (-1)^{a+k+2}, r_2] \dots \langle (-1)^{a+k+u}, r_u]$ with $1 \leq r_u < \cdots < r_1 \leq n$;*
- *if $p > 0$: $\; n+1 \geq i_1 > ... > i_{p-1} > |i_p|$;*
- *if $p > 0$ and $i_p \leq 0$: $k = 0$, $w_r = 1$ and $i_p \neq -n$;*
- *if $k = 0$ and $i_p > 0$: w_r is of form (9) with $|m_1| < i_p$.*

The affine length of w of the first (resp. second) type is $k+1$ (resp. $p+k$) and we have $0 \leq p \leq n+1$.

Now suppose that $L(w) = 1$, then w has a unique reduced expression of the form:

$$\langle i, n]\; t_{n+1}\; v \tag{13}$$

where $-n \leq i \leq n+1$ and $v \in W^c(B_{n+1})$ and:

- *if $1 < i \leq n+1$ then v is of the form (9) such that for $1 \leq j \leq r$ either $m_j = n - j + 1$ or $|m_j| < i$;*
- *if $i \leq 0$ and $i \neq -1$ then $v = \langle h, n]^{-1}$ with $-n \leq h \leq n+1$;*
- *if $i = (-1)^a$, then either $v = \langle h, n]^{-1}$ for $-n \leq h \leq n+1$, or*

$$v = \langle (-1)^{a+1}, r_1]\langle (-1)^{a+2}, r_2] \dots \langle (-1)^{a+u}, r_u] \text{ with } 1 \le r_u < \dots < r_1 \le n.$$

Conversely, every w of the above form is in $W^c(\tilde{B}_{n+1})$.

Proof. We start with an element $w \in W^c(\tilde{B}_{n+1})$ with $L(w) = m \geq 2$, written as in (10). Lemma 4.6 provides us with a good part of the Theorem. Indeed we already know that the sequence $i_1, \dots, i_m$ either stops at the first non-positive integer or stabilizes as a sequence of alternating 1 and -1 or as a sequence of $-n$, whence the classification into first and second type. So we just have to deal with the final term w_r, with the same kind of arguments as in the proof of Lemma 4.6.

For an element of first type, any reduced expression of w_r must start with σ_n, actually it must be some right truncation of $\langle -n, n]$, otherwise we would produce a braid and full commutativity would fail. The easiest way to write it is $(\langle f, n])^{-1}$.

For an element of second type with $k > 0$, any reduced expression of w_r must start with σ_1 if $j_k = -1$ or $\sigma_{\bar{1}}$ if $j_k = 1$. Writing $w_r = \langle m_1, n_1]\langle m_2, n_2] \dots \langle m_u, n_u]$ in form (9) we see from the conditions of Theorem 4.2 that indeed $m_s = (-1)^{a+k+s}$.

The fact that $w_r = 1$ for $p > 0$ and $i_p \leq 0$ is stated in Lemma 4.6. Now if $k = 0$ and $i_p > 0$ we write again w_r in form (9) and check that the only additional condition is $|m_1| < i_p$.

For an element $w = \langle i, n]\ t_{n+1},\ v \in W^c(\tilde{B}_{n+1})$ of affine length 1, the arguments are similar, using form (9) for v. If $i < -1$ or $i = 0$, any reduced expression of v must start with σ_n on the left and must be some right truncation of $\langle -n, n]$, that is, some $\langle h, n]^{-1}$. If $i = \pm 1$, such elements $\langle h, n]^{-1}$ are suitable, as well as those in form (6) that start with σ_1 if $i = -1$ or $\sigma_{\bar{1}}$ if $i = 1$, whence the result as previously.

Assume now that $i > 1$ and write $v = \langle m_1, n_1]\langle m_2, n_2] \dots \langle m_u, n_u]$ in form (9). If $i = n + 1$, there is no further condition on v, while if $i \leq n$, any reduced expression of v must start with σ_n or with σ_t with $t < i$.

The fact that any element of one of these forms is fully commutative is proven by an easy induction. $\square$

We remark that elements of the first type and elements of affine length 1 of the form $\langle i, n]\ t_{n+1}\langle h, n]^{-1}$ have a unique reduced expression. Moreover, an element of affine length at least 2 has a unique reduced expression if and only if it is of the first type. Inserting the elements of affine length 1 in the first type and second type sets would not have given us a partition of the set of those elements, as we can see in the example of $W^c(\tilde{B}_4)$. This is the reason why we handle them separately.

Remark 4.8. We did not include $W^c(\tilde{B}_3)$ in the theorem of classification because it is clear from the pushing algorithm in the Theorem that $W(\tilde{B}_n)$ is an "affinization" of $W(D_n)$ for $n \geq 4$, which is not the case of $W(\tilde{B}_3)$ which can be looked at as an affinisation of $W(B_2)$ or $W(A_3)$. The classification of elements in $W^c(\tilde{B}_3)$ is fairly easy, we leave it to the reader by recommending to consider our algorithm of pushing while seeing it as an affinisation of $W(B_2)$.

4.3. **Reduced expressions of elements of $W^c(\tilde{D}_{n+1})$.** By ψ_1 (resp. ψ_n) we denote the automorphism of $W(\tilde{D}_{n+1})$ that exchanges $\sigma_{\bar{1}}$ with σ_1 (resp. $\sigma_{\bar{n}}$ with σ_n) and fixes the other generators.

Lemma 4.9. *Let w be a fully commutative element in $W(\tilde{D}_{n+1})$ with $L(w) = m \geq 2$. Fix a reduced expression of w as follows:*

$$w = u_1\sigma_{\bar{n}}u_2\sigma_{\bar{n}}\dots u_m\sigma_{\bar{n}}u_{m+1}$$

with u_i, for $1 \leq i \leq m+1$, a reduced expression of a fully commutative element in $W^c(D_{n+1})$. Then $u_2, \dots, u_m$ are $\tilde{D}$-extremal elements, and there is a reduced expression of w of the form:

$$w = \langle i_1, n]\langle j_1, n-1]\sigma_{\bar{n}}\langle i_2, n]\langle j_2, n-1]\sigma_{\bar{n}}\dots\langle i_m, n]\langle j_m, n-1]\sigma_{\bar{n}}v_{m+1} \tag{14}$$

where $v_{m+1} \in W^c(D_{n+1})$, $i_1 = n+1$ or $|i_1| \leq n$, $j_1 = n$ or $|j_1| \leq n-1$, $|j_1| < i_1$ or $j_1 = -i_1 = \pm 1$, and for $2 \leq t \leq m$ the elements $\langle i_t, n]\langle j_t, n-1]$ belong to the following list:

$$\begin{aligned} &\textbf{(a)} \quad \langle n+1, n]\langle -(n-1), n-1] = \langle -(n-1), n-1], \\ &\textbf{(b)} \quad \langle n, n]\langle -(n-1), n-1] = \sigma_n\langle -(n-1), n-1], \\ &\textbf{(c)} \quad \langle i, n]\langle j, n-1] \textit{ with } 2 \leq i \leq n-1 \textit{ and } |j| < i, \\ &\textbf{(d)} \quad \langle i, n]\langle j, n-1] \textit{ with } i = -j = \pm 1. \end{aligned} \tag{15}$$

Proof. Since $\sigma_{\bar{n}}$ commutes with all the generators but σ_{n-1}, every two consecutive occurrences of $\sigma_{\bar{n}}$ must have in between an occurrence of σ_{n-1}, for the sake of keeping the expression reduced, and this occurrence must be properly more than once, for the sake of full commutativity; this shows that the u_i with $2 \leq i \leq m$ must be $\tilde{D}$-extremal.

Now we proceed as in the proof of Lemma 4.6: we use form (9) for u_1 and push the terms $\langle m_i, n_i]$ with $n_i < n-1$ to the right of the leftmost $\sigma_{\bar{n}}$ (as well as the term σ_n if $n_1 = n$ and $n_2 < n-1$, so that the term $\langle i_1, n]\langle j_1, n-1]$, if non trivial, has $|j_1| \leq n-1$), then we repeat this procedure with the new u_2 thus obtained and the $\sigma_{\bar{n}}$ appended at its right, and so on, until we get formally form (14). For the rest it is sufficient to see Definition 4.4 and notice that, in the $\tilde{D}$-extremal element

$\langle -(n-1), n] = \langle -(n-1), n-1]\sigma_n$, the rightmost term σ_n can also be pushed to the right of the $\sigma_{\bar{n}}$ that follows it on its right, so that this element is not needed in the list. $\square$

Lemma 4.10. *Let w be a fully commutative element in $W(\tilde{D}_{n+1})$ with $L(w) = m \geq 2$. We write w in form (14) from Lemma 4.9. For $2 \leq t \leq m$ we have either $i_t > 0$ or $i_t = -1$. Furthermore:*

(1) If $j_t > 1$ and $t \leq m-1$, then $|i_{t+1}| < j_t < i_t$.
(2) If $-(n-1) < j_t < -1$ or $j_t = 0$, then $t = m$ and $v_{m+1} = 1$.
(3) If $j_t = \pm 1$ then $i_s = -j_t = -j_s$ for all $s > t$, $s \leq m$.
(4) If $j_t = -(n-1)$ then $j_s = -(n-1)$ for $2 \leq s \leq m$ and $i_s = n$ for $2 < s \leq m$ while $i_2 = n$ or $n+1$.

Proof. We refer to the element $v_t = \langle i_t, n]\langle j_t, n-1]$ according to list (15) in Lemma 4.9. We remark from that list that either $i_t > 0$ or $i_t = -1$. We call an expression $\sigma_{a_1} \dots \sigma_{a_u}$ *admissible* if it is the reduced expression of a fully commutative element.

In case (1) the element v_t has form (c) with $j > 1$. Now $\langle i, n]\langle j, n-1]\sigma_{\bar{n}}\sigma_s$ is admissible if and only if $s < j$ or $\sigma_s = \sigma_{\bar{1}}$, since as before: σ_n (resp. σ_{n-1}, resp. σ_u with $j \leq u < n-1$) would produce the braid $\sigma_n\sigma_{n-1}\sigma_n$ (resp. $\sigma_{n-1}\sigma_{\bar{n}}\sigma_{n-1}$, resp. $\sigma_u\sigma_{u+1}\sigma_u$).

In case (2) the element v_t has form (c) with $-(n-1) < j < -1$ or $j = 0$. There is no generator σ_s that can make $\langle i, n]\langle j, n-1]\sigma_{\bar{n}}\sigma_s$ admissible.

In case (3) indeed the element v_t has form (c) or (d). Again $\sigma_n\langle -1, n-1]\sigma_{\bar{n}}\sigma_s$ is admissible if and only if $\sigma_s = \sigma_1$. Applying ψ_1 gives the other case.

In case (4) the element v_t has form (a) or (b). We note that $\langle -(n-1), n-1]\sigma_{\bar{n}}\sigma_s$ is admissible if and only if $\sigma_s = \sigma_n$, so the only choice for v_{t+1} is form (b). We now look for possible forms of v_{t-1} using the previous cases : forms (c) and (d) cannot be followed by form (a) or (b); forms (a) and (b) are followed by (b). Hence v_2 may have form (a) while v_s for $2 < s \leq m$ must have form (b). $\square$

These lemmas are the key to the normal form of fully commutative elements in $W(\tilde{D}_{n+1})$.

Theorem 4.11. *Let $w \in W^c(\tilde{D}_{n+1})$ with $L(w) \geq 2$. Then w can be written in a unique way as a reduced word of one and only one of the following two forms, for non negative integers p and k:*

First type:

$$w = (\sigma_{\bar{n}})^{\epsilon}\sigma_n^{\eta}\langle i, n-1]\sigma_{\bar{n}}(\sigma_n\langle -(n-1), n-1]\ \sigma_{\bar{n}})^k\ \langle f, n]^{-1} \tag{16}$$

with $-n \leq f \leq n+1$, $-(n-1) \leq i \leq n$, $\epsilon = 0$ or 1, $\eta = 0$ or 1, $\epsilon\eta = 0$ and if $\epsilon + \eta > 0$ then $i = -(n-1)$. In other words the term $(\sigma_n\langle -(n-1), n-1]\ \sigma_{\bar{n}})^k$ may be preceded either by $\sigma_n\langle -(n-1), n-1]\sigma_{\bar{n}}$ or

$\langle i, n-1]\sigma_{\bar{n}}$ with $-(n-1) \le i \le n$, or by $\sigma_{\bar{n}}\langle -(n-1), n-1]\sigma_{\bar{n}}$.

Second type:

(17)

$$w = \langle i_1, n]\langle j_1, n-1]\ \sigma_{\bar{n}} \dots \langle i_p, n]\langle j_p, n-1]\ \sigma_{\bar{n}}(\langle -1, n]\langle 1, n-1]\ \sigma_{\bar{n}})^k\ w_r \quad \text{if } p > 0,$$

$$w = (\langle -1, n]\langle 1, n-1]\ \sigma_{\bar{n}})^k\ w_r \quad \text{if } p = 0,$$

with $w_r \in W^c(D_{n+1})$ and

- *if $p > 0$: $n+1 \ge i_1 > j_1 > i_2 > j_2 ... > i_p > |j_p| \ge 0, i_p > 1, j_p \ne -1$;*
- *if $p > 0$ and $j_p \le 0$: $k = 0$, $w_r = 1$ and $j_p \ne -(n-1)$;*
- *if $k = 0$ and $j_p > 1$: w_r is of form (9) such that $|m_1| < j_p$;*
- *if $k > 0$ or if $k = 0$ and $j_p = 1$: $w_r = \langle -1, r_1]\langle 1, r_2] \dots \langle (-1)^u, r_u]$ with $1 \le r_u < \cdots < r_1 \le n$ and $1, -1$ alternate;*

or w is the image under ψ_1 of such an element (in this case one must replace the condition $j_p \ne -1$ by $j_p \ne 1$).

The affine length of w of the first (resp. second) type is $k+1+\epsilon$ (resp. $p+k$) and we have $0 \le p \le \frac{n-1}{2}$.

Now suppose that $L(w) = 1$, then it has a unique reduced expression of the form:

(18) $\qquad \langle i, n]\langle j, n-1]\sigma_{\bar{n}}\ v \qquad$ *where $v \in W^c(D_{n+1})$ has form (9) and*

- *either $i = n+1$, $j = n$ and v is arbitrary,*
- *or $|j| \le n-1$, $i = -1$ or $1 \le i \le n+1$, $|j| < i$ or $j = -i = \pm 1$, and:*
  - *if $|j| = 1$ and $i \le n$ then $\mathscr{L}(v) \subseteq \{\sigma_{-j}\}$ (with the convention $\sigma_{-1} = \sigma_{\bar{1}}$);*
  - *if $|j| = 1$ and $i = n+1$ then $\mathscr{L}(v) \subseteq \{\sigma_{-j}, \sigma_n\}$;*
  - *if $j < -1$ or $j = 0$, and $i \le n$, then $v = 1$;*
  - *if $j < -1$ or $j = 0$, and $i = n+1$, then $v = \langle f, n]^{-1}$ with $-n \le f \le n+1$;*
  - *if $j > 1$ and $i \le n$ then $|m_1| < j$;*
  - *if $j > 1$ and $i = n+1$ then either $|m_1| < j$ or, for some $s \le r$, $m_1 = n_1 = n, \dots, m_s = n_s = n-s+1$, $|m_{s+1}| < j$.*

Conversely, every w of the above form is in $W^c(\tilde{D}_{n+1})$.

Proof. Let w be a fully commutative element in $W(\tilde{D}_{n+1})$ with $L(w) = m \ge 2$, written in form (14) from Lemma 4.9. We consider the terms $\langle i_t, n]\langle j_t, n-1]$ in w for $2 \le t \le m$.

If one of them has form (a) or (b), we know from Lemma 4.10 (4) that all of them have form (b) except possibly the first one that can have form (a), we thus get some power of $\sigma_n\langle -(n-1), n-1]\ \sigma_{\bar{n}}$: this is the first type. Same arguments as in the proof of this lemma show that the rightmost term v_{m+1} has to be some right truncation of $\sigma_n\langle -(n-1), n-1]\sigma_n$, more easily written in the form $\langle f, n]^{-1}$ with $-n \leq f \leq n+1$. We now check the conditions on the introductory term $v_1 = \langle i_1, n]\langle j_1, n-1]$.

If the term with $t = 2$ has form (b) we have to find conditions for the expression $v_1\sigma_{\bar{n}}\sigma_n\langle -(n-1), n-1]$ to be admissible. If non trivial, v_1 ends with σ_{n-1} by the assumption $|j_1| < i_1$ and if it contains a σ_n (i.e. $|i_1| \leq n$) we must have $j_1 = -(n-1)$ to forbid the braid $\sigma_n\sigma_{n-1}\sigma_n$ (remember that $\sigma_{\bar{n}}$ and σ_n commute). So there is no choice but $i_1 = n$. We thus have a left truncation of $\sigma_n\langle -(n-1), n-1]$.

If the term with $t = 2$ has form (a) we inspect $v_1\sigma_{\bar{n}}\langle -(n-1), n-1]$. If non trivial, v_1 must end with σ_n, which is against our convention in Lemma 4.9, so v_1 is trivial, hence the form for the first type.

Otherwise those terms have form (c) or (d) and we know from Lemma 4.10 (1) that at most the $[n/2]$ first terms may have form (c), so eventually, for t big enough, all the terms will have form (d), i.e. we get some power of $\langle -1, n]\langle 1, n-1]\ \sigma_{\bar{n}}$ or of its image under ψ_1: this is the second type. We check first the form of v_{m+1}. The proof of Lemma 4.10 (3), resp. (2), resp. (1), gives the case where $k > 0$, resp. $k = 0$ and $j_p \leq 0$, resp. $k = 0$ and $j_p > 0$. The form of the introductory term $\langle i_1, n]\langle j_1, n-1]\ \sigma_{\bar{n}} \dots \langle i_p, n]\langle j_p, n-1]\ \sigma_{\bar{n}}$ is an immediate consequence of the previous lemmas.

We have $n + 1 \geq i_1 \geq i_p + 2p \geq 2 + 2p$ hence $p \leq \frac{n-1}{2}$.

If w has affine length 1, as in the proof of Lemma 4.9, we use the fact that $\sigma_{\bar{n}}$ commutes with all generators except σ_{n-1} to obtain the expression $\langle i, n]\langle j, n-1]\sigma_{\bar{n}}v$ where either $|j| \leq n-1$ or $j = n$ and $i = n+1$. The various cases are obtained in the same way as previously, we do not detail this. $\square$

We note that the element of affine length 2

$$\sigma_{\bar{n}}\ \langle -(n-1), n-1]\ \sigma_{\bar{n}}\ \langle f, n]^{-1}$$

is a first type element. Inserting the elements of affine length 1 in the first type and second type sets would not have given us a partition of the set of those elements as we can see when listing elements in $W^c(\tilde{D}_4)$. This is the reason why we handle them separately.

5. The towers of fully commutative elements

When a Coxeter group is a parabolic subgroup of another, full commutativity is preserved by the natural injection. This is not the case for the embeddings defined in

Corollary 2.2: the monomorphism $G_n : W(\tilde{D}_n) \longrightarrow W(\tilde{D}_{n+1})$ does not preserve full commutativity, even for the simple reflexion σ_{n-1}^-, while $E_n : W(\tilde{B}_n) \longrightarrow W(\tilde{B}_{n+1})$ preserves the full commutativity of first type elements and elements of affine length 1 in $W^c(\tilde{B}_n)$, but does not preserve it for $t_n\sigma_{n-1}t_n$, for example, in the set of second type fully commutative elements. We will use the normal form for fully commutative elements established in Theorem 4.11 (resp. Theorem 4.7) to produce embeddings from $W^c(\tilde{B}_n)$ into $W^c(\tilde{B}_{n+1})$ and from $W^c(\tilde{D}_n)$ into $W^c(\tilde{D}_{n+1})$.

5.1. **The tower of $\tilde{B}$-type fully commutative elements.** For $n > 2$, we denote by $W_1^c(\tilde{B}_n)$ the set of first type fully commutative elements in addition to fully commutative elements of affine length 1, and by $W_2^c(\tilde{B}_n)$ the set of second type fully commutative elements. We thus have the following partition (under the convention $D_3 := A_3$):

$$W^c(\tilde{B}_n) = W_1^c(\tilde{B}_n) \bigsqcup W_2^c(\tilde{B}_n) \bigsqcup W^c(D_n).$$

Definition 5.1. *For any $w \in W^c(\tilde{B}_n)$ we define elements $I(w)$ and $J(w)$ of $W(\tilde{B}_{n+1})$ by the following expressions:*

- *if $w \in W_2^c(\tilde{B}_n)$, then $I(w)$ (resp. $J(w)$) is obtained by substituting $\sigma_n t_{n+1}$ (resp. $t_{n+1}\sigma_n$) to t_n in the normal form (12) for w;*
- *if $w \in W_1^c(\tilde{B}_n)$, then $I(w) = J(w)$ is obtained by substituting $\sigma_n t_{n+1}\sigma_n$ to t_n in the normal form (11) or (13) for w;*
- *if $w \in W^c(D_n)$, then $I(w) = J(w) = w$.*

Theorem 5.2. *For any $w \in W^c(\tilde{B}_n)$, the expressions for $I(w)$ and $J(w)$ in Definition 5.1 are reduced and they are reduced expressions for fully commutative elements in $W(\tilde{B}_{n+1})$. The maps thus defined:*

$$I, J : W^c(\tilde{B}_n) \longrightarrow W^c(\tilde{B}_{n+1})$$

are injective, preserve the affine length and satisfy

$$l(I(w)) = l(J(w)) = l(w) + L(w) \quad \text{for } w \in W_2^c(\tilde{B}_n),$$
$$l(I(w)) = l(J(w)) = l(w) + 2L(w) \quad \text{for } w \in W_1^c(\tilde{B}_n).$$

The injections I and J map first type (resp. second type) elements to first type (resp. second type) elements and their images intersect exactly on $I(W_1^c(\tilde{B}_n) \bigsqcup W^c(D_n))$.

Proof. The proof is mutatis mutandis the proof of [4, Theorem 5.2]. We will only emphasize the main fact, which must be kept in mind for the applications. For an element w of the second type, the expression of $J(w)$ is obtained by substituting

$t_{n+1}\sigma_n$ to t_n in the normal form for w. In order to find the normal form of $J(w)$ one uses the property that t_{n+1} commutes with $\sigma_{\bar{1}}$ and σ_i for $1 \leq i < n$, hence each t_{n+1} can be pushed to its left until either it is stopped by a σ_n or it is in the leftmost position. In particular, the leftmost term in the normal form of $J(w)$ is t_{n+1} whereas no reduced expression of $I(w)$ can have t_{n+1} as its leftmost term (since the expression $t_{n+1}I(w)$ is also a normal form of the second type). Consequently the images $I(W_2^c(\tilde{B}_n))$ and $J(W_2^c(\tilde{B}_n))$ are disjoint. $\square$

We remark that the injections I and J on $W_1^c(\tilde{C}_n) \bigsqcup W^c(B_n)$ are but the restriction of E_n in (2). Actually I and J may be defined on all $W(\tilde{B}_n)$, but as we don't need this, we won't examine it further.

5.2. **The tower of $\tilde{D}$-type fully commutative elements.** For $n > 3$, we denote by $W_1^c(\tilde{D}_n)$ the set of first type fully commutative elements in addition to fully commutative elements of affine length 1, and by $W_2^c(\tilde{D}_n)$ the set of second type fully commutative elements. We thus have the following partition:

$$W^c(\tilde{D}_n) = W_1^c(\tilde{D}_n) \bigsqcup W_2^c(\tilde{D}_n) \bigsqcup W^c(D_n).$$

Definition 5.3. *For any $w \in W^c(\tilde{D}_n)$ we define elements $I(w)$ and $J(w)$ of $W(\tilde{D}_{n+1})$ by the following expressions:*

- *if $w \in W_2^c(\tilde{D}_n)$, then $I(w)$ (resp. $J(w)$) is obtained by substituting $\sigma_n\sigma_{n-1}\sigma_{\bar{n}}$ (resp. $\sigma_{\bar{n}}\sigma_{n-1}\sigma_n$) to $\sigma_{\bar{n-1}}$ in the normal form (17) for w;*
- *if $w \in W_1^c(\tilde{D}_n)$, then $I(w) = J(w)$ is obtained by substituting $\sigma_{n-1}\sigma_{\bar{n}}\sigma_n$ to $\sigma_{\bar{n-1}}$ in the normal form (16) or (18) for w, with the following exceptions:*
  - *if w is a first type element with $\epsilon = 1$ then the leftmost term $\sigma_{\bar{n-1}}$ in the normal form (16) must be substituted by $\sigma_n\sigma_{\bar{n}}\sigma_{n-1}$;*
  - *if w has affine length 1 with $i \neq n$, then $\sigma_{\bar{n-1}}$ in (18) must be substituted by $\sigma_n\sigma_{n-1}\sigma_{\bar{n}}$.*
- *if $w \in W^c(D_n)$, then $I(w) = J(w) = w$.*

Theorem 5.4. *For any $w \in W^c(\tilde{D}_n)$, the expressions for $I(w)$ and $J(w)$ in Definition 5.3 are reduced and they are reduced expressions for fully commutative elements in $W(\tilde{D}_{n+1})$. The maps thus defined:*

$$I, J : W^c(\tilde{D}_n) \longrightarrow W^c(\tilde{D}_{n+1})$$

are injective, preserve the affine length and satisfy

$$l(I(w)) = l(J(w)) = l(w) + 2L(w).$$

The injections I and J map first type (resp. second type) elements to first type (resp. second type) elements and their images intersect exactly on $I(W_1^c(\tilde{D}_n) \bigsqcup W^c(D_n))$.

Proof. We start with $w \in W_2^c(\tilde{D}_n)$ given in form (17) and notice that, since σ_n commutes with all generators except σ_{n-1}, we have :

$$I(\langle i_s, n-1]\langle j_s, n-2]\ \sigma_{\bar{n-1}}) = \langle i_s, n]\langle j_s, n-1]\ \sigma_{\bar{n}}$$

so that $I(w)$ is indeed the reduced expression of an element in $W_2^c(\tilde{D}_{n+1})$ with the same parameters $i_1, \ldots, i_p, j_1, \ldots, j_p, k, w_r$, as w. On the other hand, again using commutation relations, we find:

$$\begin{aligned} &J(\langle i_{s-1}, n-1]\langle j_{s-1}, n-2]\ \sigma_{\bar{n-1}}\langle i_s, n-1]\langle j_s, n-2]\ \sigma_{\bar{n-1}}) \\ &\quad = \langle i_{s-1}, n-1]\sigma_{\bar{n}}\langle j_{s-1}, n]\ \langle i_s, n-1]\sigma_{\bar{n}}\langle j_s, n]. \end{aligned}$$

Hence $J(w)$ is also the reduced expression of an element in $W_2^c(\tilde{D}_{n+1})$, here with a shift in parameters: if we denote with primes the parameters for $J(w)$ we have

$$i'_1 = n+1, j'_1 = i_1, i'_2 = j_1, j'_2 = i_2, \ldots, j'_p = i_p, w'_r = \langle x, n]w_r,$$

with $x = j_p$ if $k = 0$, or $x = \pm 1$, so $x \neq n+1$. In particular this shift acts on the sequence of 1 and -1, so $J(w)$ will be the image under ψ_1 of an element in form (17).

The injectivity of both maps is clear from the unicity of the normal form, since the parameters of the images determine the parameters of the source element. Furthermore we remark that in the normal form for $I(w)$, the term on the right of the rightmost $\sigma_{\bar{n}}$ is w_r which belongs to $W^c(D_n)$ whereas for $J(w)$ the corresponding term is $\langle x, n]w_r$ which contains σ_n, so the images of I and J on second type elements do not intersect.

We now turn to first type elements and take w given in form (16). Here

$$I(\langle i, n-2]\ \sigma_{\bar{n-1}}\sigma_{n-1}\langle -(n-2), n-2]\sigma_{\bar{n-1}}) = \langle i, n-1]\sigma_{\bar{n}}\sigma_n\langle -(n-1), n-1]\sigma_{\bar{n}}\sigma_n$$

so, if $\epsilon = 0$, $I(w)$ is the reduced expression of a first type element, with the same parameters ϵ, η, i, k, f if $\eta = 0$, while if $\eta = 1$ (hence $i = -(n-2)$) the parameters of $I(w)$, written with a prime, are $\epsilon' = \epsilon, \eta' = 0, i' = -(n-1), k' = k, f' = f$. If $\epsilon = 1$, our element starts with $\sigma_{\bar{n-1}}\langle -(n-2), n-2]\sigma_{\bar{n-1}}$ and the previous substitution for the leftmost $\sigma_{\bar{n-1}}$ would produce, again for commutation reasons, the braid $\sigma_n\sigma_{n-1}\sigma_n$. We substitute instead $\sigma_{\bar{n}}\sigma_n\sigma_{n-1}$, getting:

$$I(\ \sigma_{\bar{n-1}}\langle -(n-2), n-2]\sigma_{\bar{n-1}}\sigma_{n-1}\cdots) = \sigma_{\bar{n}}\sigma_n\langle -(n-1), n-1]\sigma_{\bar{n}}\sigma_n\sigma_{n-1}\cdots$$

hence a first type element with parameters $\epsilon' = \eta' = 0$, $i' = n$, $k' = k+1$ and $f' = f$. The set of such parameters is disjoint from the set obtained with $\epsilon = 0$, indeed for those we had $i' \leq n-1$, injectivity for first type follows.

The last case to consider is elements of affine length 1. Here substituting $\sigma_{n-1}\sigma_{\bar{n}}\sigma_n$ to $\sigma_{n\bar{-}1}$ in the normal form (16) transforms $\langle i, n-1]\langle j, n-2]\sigma_{n\bar{-}1}\ v$ into $\langle i, n-1]\langle j, n-1]\sigma_{\bar{n}}\sigma_n\ v$ which is indeed, if $i = n$, the reduced expression of a fully commutative element in the required form (16), with parameters $i' = n+1$, $j' = j$ and $v' = \sigma_n v$. If $i < n$ the correct substitution is $\sigma_n\sigma_{n-1}\sigma_{\bar{n}}$, resulting in $\langle i, n]\langle j, n-1]\sigma_{\bar{n}}\ v$ that again has the required form, with parameters $i' = i$, $j' = j$ and $v' = v$. The other assertions follow as previously, in particular injectivity follows from the fact that the parameters i, j and v determine a fully commutative element of affine length 1. $\square$

6. Faithfulness of towers of Temperley-Lieb algebras

Let K be an integral domain of characteristic 0 and let q be an invertible element in K. Let Γ be a Coxeter graph with associated Coxeter system $(W(\Gamma), S)$ and Hecke algebra $H\Gamma(q)$. Following Graham [8, Definition 6.1], we define the Γ-type Temperley-Lieb algebra $TL\Gamma(q)$ to be the quotient of the Hecke algebra $H\Gamma(q)$ by the two-sided ideal generated by the elements $E_{s,t} = \sum_{w \in <s,t>} g_w$, where s and t are non commuting elements in S such that st has finite order. For w in $W(\Gamma)$ we denote by T_w the image of $g_w \in H\Gamma(q)$ under the canonical surjection from $H\Gamma(q)$ onto $TL\Gamma(q)$. The set $\{T_w \mid w \in W^c(\Gamma)\}$ forms a K-basis for $TL\Gamma(q)$ [8, Theorem 6.2].

For x, y in a given ring with identity, we define:

$$\begin{aligned} V(x,y) &= xyx + xy + yx + x + y + 1, \\ Z(x,y) &= xyxy + xyx + yxy + xy + yx + x + y + 1. \end{aligned}$$

6.1. Faithfulness of the tower of $\tilde{B}$-type Temperley-Lieb algebras.

For $n \geq 2$, the $\tilde{B}$-type Temperley-Lieb algebra with $n+2$ generators $TL\tilde{B}_{n+1}(q)$ is given by the set of generators $\left\{T_{\sigma_{\bar{1}}}, T_{\sigma_1}, \dots T_{\sigma_n}, T_{t_{n+1}}\right\}$, with the defining relations:

$$
\left\{
\begin{array}{l}
T_{\sigma_i}T_{\sigma_j} = T_{\sigma_j}T_{\sigma_i} \text{ for } 1 \leq i,j \leq n \text{ and } |i-j| \geq 2,\\
T_{\sigma_i}T_{t_{n+1}} = T_{t_{n+1}}T_{\sigma_i} \text{ for } 1 \leq i \leq n-1,\\
T_{\sigma_i}T_{\sigma_{i+1}}T_{\sigma_i} = T_{\sigma_{i+1}}T_{\sigma_i}T_{\sigma_{i+1}} \text{ for } 1 \leq i \leq n-1,\\
T_{\sigma_{\bar{1}}}T_{\sigma_2}T_{\sigma_{\bar{1}}} = T_{\sigma_2}T_{\sigma_{\bar{1}}}T_{\sigma_2},\\
T_{\sigma_{\bar{1}}}T_{t_{n+1}} = T_{t_{n+1}}T_{\sigma_{\bar{1}}} \text{ and } T_{\sigma_{\bar{1}}}T_{\sigma_i} = T_{\sigma_i}T_{\sigma_{\bar{1}}} \text{ for } 1 \leq i \leq n, i \neq 2,\\
T_{t_{n+1}}T_{\sigma_n}T_{t_{n+1}}T_{\sigma_n} = T_{\sigma_n}T_{t_{n+1}}T_{\sigma_n}T_{t_{n+1}},\\
T^2 = (q-1)T + q \text{ for } T \in \left\{T_{\sigma_{\bar{1}}}, T_{\sigma_1}, \dots T_{\sigma_n}, T_{t_{n+1}}\right\},\\
V(T_{\sigma_{\bar{1}}}, T_{\sigma_2}) = 0 \text{ and } V(T_{\sigma_i}, T_{\sigma_{i+1}}) = 0 \text{ for } 1 \leq i \leq n-1,\\
Z(T_{\sigma_n}, T_{t_{n+1}}) = 0.
\end{array}
\right. \tag{19}
$$

We set $TL\tilde{B}_2(q) = K$. We temporarily denote by h_w, $w \in W^c(\tilde{B}_n)$, the basis elements of $TL\tilde{B}_n(q)$ to distinguish them from those of $TL\tilde{B}_{n+1}(q)$.

Lemma 6.1. *The morphism of algebras $Q_n : H\tilde{B}_n(q) \longrightarrow H\tilde{B}_{n+1}(q)$ defined in (7) induces the following morphism of algebras, which we also denote by Q_n:*

$$
\begin{aligned}
Q_n : TL\tilde{B}_n(q) &\longrightarrow TL\tilde{B}_{n+1}(q)\\
h_{\sigma_i} &\longmapsto T_{\sigma_i} \quad \textit{for } 1 \leq i \leq n-1\\
h_{\sigma_{\bar{1}}} &\longmapsto T_{\sigma_{\bar{1}}}\\
h_{t_n} &\longmapsto T_{\sigma_n}T_{t_{n+1}}T_{\sigma_n}^{-1}.
\end{aligned}
$$

The restriction of Q_n to $TLD_n(q)$ is an injective morphism into $TLD_{n+1}(q)$ and satisfies $Q_n(h_w) = g_{I(w)} = g_{J(w)}$ for $w \in W^c(D_n)$.

Proof. See [4, Lemma 6.1]. □

The aim of this section is to show, using the normal form of Theorem 4.7, that the morphism Q_n is an injection. We set $p = 1/q$, so that we have:

$$Q_n(h_{t_n}) = pT_{\sigma_n t_{n+1}\sigma_n} + (p-1)T_{\sigma_n t_{n+1}}. \tag{20}$$

Lemma 6.2. *If $w \in W_1^c(\tilde{B}_n) \bigsqcup W^c(D_n)$ we have:*

$$Q_n(h_w) = p^{L(w)}T_{I(w)} + \sum_{\substack{L(x)\leq L(w)\\ l(x)<l(I(w))}} \alpha_x T_x \qquad (\alpha_x \in K).$$

Proof. Similar to [4, Lemma 6.2]. □

Proposition 6.3. *Let w be in $W_2^c(\tilde{B}_n)$, then for some $\alpha_x, \beta_y \in K$ we have*

$$Q_n(h_w) = (-1)^{L(w)}T_{I(w)} + (-p)^{L(w)}T_{J(w)} + \sum_{\substack{L(y)=L(w)\\ l(y)<l(I(w))}} \beta_y T_y + \sum_{L(x)<L(I(w))} \alpha_x T_x.$$

Proof. Here also the proof is similar to the proof of [4, Proposition 6.3] for type $\tilde{C}$. Indeed the proof in *loc.cit.* relies on the relations involving $T_{\sigma_{n-1}}$, T_{σ_n} and $T_{t_{n+1}}$, that are the same in $TL\tilde{B}_{n+1}(q)$ and $TL\tilde{C}_{n+1}(q)$, and on the fact that $T_{t_{n+1}}$ commutes with $R_n(TLB_n(q))$, here replaced by $Q_n(TLD_n(q))$. We mention the following more precise statement from *loc.cit.*:

Let w be in $W_2^c(\tilde{B}_n)$, whose normal form (12) begins and ends with t_n. Then for some $\alpha_x, \beta_y \in K$ we have:

$$\begin{aligned} Q_n(h_w) = (-1)^{L(w)}T_{I(w)} &+ (-p)^{L(w)}T_{J(w)} \\ &+ T_{t_{n+1}}T_{\sigma_n}T_{t_{n+1}}\,(\sum_{\substack{L(y)=L(w)-2\\ l(y)<l(I(w))-3}}\beta_y T_y) + \sum_{L(x)<L(I(w))}\alpha_x T_x. \end{aligned} \tag{21}$$

□

We can conclude as in [4, Theorem 6.4], with a similar proof:

Theorem 6.4. *The tower of affine Temperley-Lieb algebras*

$$TL\tilde{B}_2(q) \xrightarrow{Q_2} TL\tilde{B}_3(q) \longrightarrow \cdots TL\tilde{B}_n(q) \xrightarrow{Q_n} TL\tilde{B}_{n+1}(q) \longrightarrow \cdots$$

is a tower of faithful arrows.

6.2. **Faithfulness of the tower of $\tilde{D}$-type Temperley-Lieb algebras.** For $n \geq 3$, the $\tilde{D}$-type Temperley-Lieb algebra with $n+2$ generators $TL\tilde{D}_{n+1}(q)$ is given by the set of generators $\left\{T_{\sigma_{\bar{1}}}, T_{\sigma_1}, \dots T_{\sigma_n}, T_{\sigma_{\bar{n}}}\right\}$, with the defining relations:

$$\left\{ \begin{array}{l} T_{\sigma_i}T_{\sigma_j} = T_{\sigma_j}T_{\sigma_i} \text{ for } 1 \leq i,j \leq n \text{ and } |i-j| \geq 2, \\ T_{\sigma_{\bar{1}}}T_{\sigma_{\bar{n}}} = T_{\sigma_{\bar{n}}}T_{\sigma_{\bar{1}}} \text{ and } T_{\sigma_{\bar{1}}}T_{\sigma_i} = T_{\sigma_i}T_{\sigma_{\bar{1}}} \text{ for } 1 \leq i \leq n, i \neq 2, \\ T_{\sigma_{\bar{n}}}T_{\sigma_i} = T_{\sigma_i}T_{\sigma_{\bar{n}}} \text{ for } 1 \leq i \leq n, i \neq n-1, \\ T_{\sigma_i}T_{\sigma_{i+1}}T_{\sigma_i} = T_{\sigma_{i+1}}T_{\sigma_i}T_{\sigma_{i+1}} \text{ for } 1 \leq i \leq n-1, \\ T_{\sigma_{\bar{1}}}T_{\sigma_2}T_{\sigma_{\bar{1}}} = T_{\sigma_2}T_{\sigma_{\bar{1}}}T_{\sigma_2} \text{ and } T_{\sigma_{\bar{n}}}T_{\sigma_{n-1}}T_{\sigma_{\bar{n}}} = T_{\sigma_{n-1}}T_{\sigma_{\bar{n}}}T_{\sigma_{n-1}}, \\ T^2 = (q-1)T + q \ \text{ for } T \in \left\{T_{\sigma_{\bar{1}}}, T_{\sigma_1}, \dots, T_{\sigma_n}, T_{\sigma_{\bar{n}}}\right\}, \\ V(T_{\sigma_{\bar{1}}}, T_{\sigma_2}) = V(T_{\sigma_{\bar{n}}}, T_{\sigma_{n-1}}) = 0, \\ V(T_{\sigma_i}, T_{\sigma_{i+1}}) = 0 \text{ for } 1 \leq i \leq n-1. \end{array} \right. \tag{22}$$

We set $TL\tilde{D}_3(q) = K$. In the following we denote by h_w, $w \in W^c(\tilde{D}_n)$, the basis elements of $TL\tilde{D}_n(q)$ to distinguish them momentarily from those of $TL\tilde{D}_{n+1}(q)$.

Lemma 6.5. *The morphism of algebras $P_n : H\tilde{D}_n(q) \longrightarrow H\tilde{D}_{n+1}(q)$ defined in (8) induces the following morphism of algebras, which we also denote by P_n:*

$$\begin{aligned} P_n : TL\tilde{D}_n(q) &\longrightarrow TL\tilde{D}_{n+1}(q) \\ h_{\sigma_i} &\longmapsto T_{\sigma_i} \quad \text{for } \sigma_i \in \{\sigma_{\bar{1}}, \sigma_1, \dots, \sigma_{n-1}\}, \\ h_{\sigma_{\bar{n-1}}} &\longmapsto T_{\sigma_n} T_{\sigma_{n-1}} T_{\sigma_{\bar{n}}} T_{\sigma_{n-1}}^{-1} T_{\sigma_n}^{-1}. \end{aligned}$$

The restriction of P_n to $TLD_n(q)$ is an injective morphism into $TLD_{n+1}(q)$ and satisfies $P_n(h_w) = g_{I(w)} = g_{J(w)}$ for $w \in W^c(D_n)$.

Proof. The lemma follows after noticing that

$$V(P_n(h_{\sigma_{\bar{n-1}}}), P_n(h_{\sigma_{n-2}})) = (T_{\sigma_n} T_{\sigma_{\bar{n}}}^{-1}) V(T_{\sigma_{n-1}}, T_{\sigma_{n-2}}) (T_{\sigma_n} T_{\sigma_{\bar{n}}}^{-1})^{-1}.$$

□

The aim of this section is to show, using the normal form of Theorem 4.11, that the morphism P_n is an injection. We set $p = 1/q$. We will use repeatedly

$$\begin{aligned} P_n(h_{\sigma_{\bar{n-1}}}) =& T_{\sigma_n\sigma_{n-1}\sigma_{\bar{n}}} + pT_{\sigma_{n-1}\sigma_{\bar{n}}\sigma_n} + pT_{\sigma_n\sigma_{\bar{n}}\sigma_{n-1}} + p^2 T_{\sigma_{\bar{n}}\sigma_{n-1}\sigma_n} \\ &+ pT_{\sigma_n\sigma_{n-1}} + p^2 T_{\sigma_{n-1}\sigma_n} + pT_{\sigma_{n-1}\sigma_{\bar{n}}} + p^2 T_{\sigma_{\bar{n}}\sigma_{n-1}} + (p^2+p) T_{\sigma_{\bar{n}}\sigma_n} \\ &+ p^2 T_{\sigma_n} + p^2 T_{\sigma_{n-1}} + p^2 T_{\sigma_{\bar{n}}} + (p^2 - p) \end{aligned} \tag{23}$$

as well as those two rules that follow from the defining relations (22):

(i) In $TL\tilde{D}_{n+1}(q)$, a product $T_w T_y$, $w, y \in W^c(\tilde{D}_{n+1})$, is either equal to T_{wy}, if $l(wy) = l(w) + l(y)$ and wy is fully commutative, or equal to a linear combination of terms T_z, $z \in W^c(\tilde{D}_{n+1})$, with $L(z) \leq L(w) + L(y)$ and $l(z) < l(w) + l(y)$.

(ii) When a braid $T_{\sigma_i} T_{\sigma_{i+1}} T_{\sigma_i}$ appears in a computation, the use of $V(T_{\sigma_i}, T_{\sigma_{i+1}}) = 0$ replaces it by a sum of terms T_z with $l(z) = 2$, 1 or 0, hence the length decreases.

Proposition 6.6. *Let $w \in W^c(\tilde{D}_n)$ be of affine length at least 2. Then for some $\alpha_x, \beta_y \in K$ we have, if w is a first type element:*

$$P_n(h_w) = p^{L(w)} T_{I(w)} + \sum_{\substack{L(x) \leq L(w) \\ l(x) < l(I(w))}} \alpha_x T_x \qquad (\alpha_x \in K),$$

and if w is a second type element:

$$P_n(h_w) = T_{I(w)} + p^{2L(w)} T_{J(w)} + \sum_{\substack{L(y) = L(w) \\ l(y) < l(I(w))}} \beta_y T_y + \sum_{L(x) < L(I(w))} \alpha_x T_x.$$

Proof. We go back to Lemma 4.9 and write w as in (14):

$$w = \langle i_1, n-1]\langle j_1, n-2]\sigma_{\bar{n-1}} \dots \langle i_m, n-1]\langle j_m, n-2]\sigma_{\bar{n-1}} v_{m+1}$$

with the notation there, in particular, for $2 \le t \le m$, the element $\langle i_t, n-1]\langle j_t, n-2]$ belongs to list (15):

(a) $\langle n, n-1]\langle -(n-2), n-2] = \langle -(n-2), n-2]$,

(b) $\langle n-1, n-1]\langle -(n-2), n-2] = \sigma_{n-1}\langle -(n-2), n-2]$,

(c) $\langle i, n-1]\langle j, n-2]$ with $2 \le i \le n-2$ and $|j| < i$,

(d) $\langle i, n-1]\langle j, n-2]$ with $i = -j = \pm 1$.

We examine the subword $x = \sigma_{\bar{n-1}}\langle i_t, n-1]\langle j_t, n-2]\sigma_{\bar{n-1}}$. When developing $P_n(h_w)$ we will develop $P_n(h_x)$ as $P_n(h_x) = P_n(h_{\sigma_{\bar{n-1}}})T_{\langle i_t,n-1]\langle j_t,n-2]}P_n(h_{\sigma_{\bar{n-1}}})$ where $P_n(h_{\sigma_{\bar{n-1}}})$ is given by (23). We are interested in terms of maximal affine length $L(w)$ and maximal Coxeter length $l(w)$, so we keep only in (23) the four terms of Coxeter length 3. We have to check the following products:

$$\begin{array}{lcl}
[1] \quad T_{\sigma_n\sigma_{n-1}\sigma_{\bar{n}}} & & [1'] \quad T_{\sigma_n\sigma_{n-1}\sigma_{\bar{n}}} \\
[2] \; p\, T_{\sigma_{n-1}\sigma_{\bar{n}}\sigma_n} & & [2'] \; p\, T_{\sigma_{n-1}\sigma_{\bar{n}}\sigma_n} \\
 & T_{\langle i_t,n-1]\langle j_t,n-2]} & \\
[3] \; p\, T_{\sigma_n\sigma_{\bar{n}}\sigma_{n-1}} & & [3'] \; p\, T_{\sigma_n\sigma_{\bar{n}}\sigma_{n-1}} \\
[4] \; p^2\, T_{\sigma_{\bar{n}}\sigma_{n-1}\sigma_n} & & [4'] \; p^2\, T_{\sigma_{\bar{n}}\sigma_{n-1}\sigma_n}
\end{array}$$

Each time a resulting subword is either non reduced or non fully commutative (i.e. containing a braid), it will contribute only to terms of strictly shorter length. We thus eliminate by inspection all possible combinations except the following : 1c1', 1d1', 2b2', 3a2', 4c4', 4d4'.

By definition, if w is a second type element, then our middle term has form (c) or (d) and we will get the maximal length elements by replacing the occurrences of $P_n(h_{\sigma_{\bar{n-1}}})$ either all by $T_{\sigma_n\sigma_{n-1}\sigma_{\bar{n}}}$, giving rise to a term $T_{I(w)}$, or all by $p^2 T_{\sigma_{\bar{n}}\sigma_{n-1}\sigma_n}$, giving rise to a term $p^{2L(w)}T_{J(w)}$.

If w is a first type element, a middle term has form (a) or (b), but form (a) can only occur if $t = 2$ by Lemma 4.10. We get the maximal length elements by replacing all occurrences of $P_n(h_{\sigma_{\bar{n-1}}})$ by $pT_{\sigma_{n-1}\sigma_{\bar{n}}\sigma_n}$, except, if form (a) occurs (for $t = 2$), the leftmost one, that must be replaced by $pT_{\sigma_n\sigma_{\bar{n}}\sigma_{n-1}}$. We indeed get $p^{L(w)}T_{I(w)}$ as the only leading term. $\square$

Lemma 6.7. *Let $w \in W^c(\tilde{D}_n)$ be of affine length 1 and write as in (18):*

$$w = \langle i, n-1]\langle j, n-2]\sigma_{\bar{n-1}}\ v \quad (v \in W^c(D_n)).$$

Let $\nu = 1$ if $i = n$ and $\nu = 0$ otherwise. If $i = n$, or if $i < n$ and $v \notin W^c(D_{n-1})$, we have:

$$P_n(h_w) = p^\nu T_{I(w)} + \sum_{\substack{L(x)=1 \\ l(x)=l(I(w)) \\ x\notin \ Im\ I}} \alpha_x T_x + \sum_{\substack{L(x)\leq 1 \\ l(x)<l(I(w))}} \alpha_x T_x \qquad (\alpha_x \in K).$$

If $i < n$ and $v \in W^c(D_{n-1})$, we have:

$$P_n(h_w) = T_{I(w)} + pT_{I(\bar{w})} + \sum_{\substack{L(x)=1 \\ l(x)=l(I(w)) \\ x\notin \ Im\ I}} \alpha_x T_x + \sum_{\substack{L(x)\leq 1 \\ l(x)<l(I(w))}} \alpha_x T_x \qquad (\alpha_x \in K)$$

where $\bar{w} = \langle i, n-2]\sigma_{\bar{n-1}}\langle j, n-1]v$.

Proof. We recall, from the proof of Theorem 5.4, that the image of I consists of elements

$$\langle k', n]\langle l', n-1]\sigma_{\bar{n}}\ u' \quad \text{with} \left\{ \begin{array}{c} \text{either } k' = n+1, l' = l, u' = \sigma_n u \text{ if } k = n, \\ \text{or } k' = k, l' = l, u' = u \text{ if } k < n, \end{array} \right. \tag{24}$$

where (k, l, u) are the parameters that determine uniquely a fully commutative element of affine length 1 and form (18). In particular elements in the image of I satisfy:

$$\begin{cases} -(n-1) \leq k' \leq n-1 \text{ or } k' = n+1, \\ -(n-2) \leq l' \leq n-1. \end{cases}$$

Let w have affine length 1 and write as in (18): $w = \langle i, n-1]\langle j, n-2]\sigma_{\bar{n-1}}\ v$ with $v \in W^c(D_n)$. We have

$$P_n(h_w) = T_{\langle i,n-1]\langle j,n-2]} P_n(h_{\sigma_{\bar{n-1}}}) T_v$$

where $P_n(h_{\sigma_{\bar{n-1}}})$ is given by (23), from which we keep the four terms of Coxeter length 3. As previously we have to check the products in the figure below to identify $T_{I(w)}$ or $pT_{I(w)}$, and possible terms among the others of the form λT_x, $\lambda \in K$ and $l(x) = l(I(w))$, and study whether x belongs to the image of I.

We write the resulting products in the form $\langle k', n]\langle l', n-1]\sigma_{\bar{n}}\ u'$ and abbreviate fully commutative as fc.

(1) If $i = n$, term [2] does provide a $pT_{I(w)}$. The other three terms may provide some T_x with $l(x) = l(I(w))$ (for instance, if $j = n-1$ and $v = 1$, the four terms have this property: this is (23)), but we will show that such x do not belong to the image of I.

$$\begin{tikzcd}[column sep=small, row sep=small]
 & {[1]\quad T_{\sigma_n\sigma_{n-1}\sigma_{\bar{n}}}} \arrow[ddr, dash] & \\
 & {[2]\ p\ T_{\sigma_{n-1}\sigma_{\bar{n}}\sigma_n}} \arrow[dr, dash] & \\
T_{\langle i,n-1]\langle j,n-2]} \arrow[uur, dash] \arrow[ur, dash] \arrow[dr, dash] \arrow[ddr, dash] & & T_v \\
 & {[3]\ p\ T_{\sigma_n\sigma_{\bar{n}}\sigma_{n-1}}} \arrow[ur, dash] & \\
 & {[4]\ p^2\ T_{\sigma_{\bar{n}}\sigma_{n-1}\sigma_n}} \arrow[uur, dash] &
\end{tikzcd}$$

For term [1], we observe that $\langle j, n-2]\sigma_n\sigma_{n-1}\sigma_{\bar{n}}v = \langle n, n]\langle j, n-1]\sigma_{\bar{n}}v$ cannot belong to the image of I since $k' = n$, impossible from the remark following (24). Similarly term [3] would correspond to $\langle j, n-2]\sigma_n\sigma_{\bar{n}}\sigma_{n-1}v = \sigma_{\bar{n}}\sigma_n\langle j, n-1]v$ that cannot belong to the image of I since $l' = n$, and term [4] would correspond to $\langle j, n-2]\sigma_{\bar{n}}\sigma_{n-1}\sigma_n v = \sigma_{\bar{n}}\langle j, n]v$ that cannot belong to the image of I for the same reason. The claim is proved in case $i = n$.

(2) If $i < n$, term [1] provides a $T_{I(w)}$. Again we must check the other three terms.

Term [2] with $j \neq -(n-2)$ gives the braid $\sigma_{n-1}\sigma_{n-2}\sigma_{n-1}$, while for $j = -(n-2)$ (hence $i = n-1$) we get $\langle -(n-1), n-1]\sigma_{\bar{n}}\sigma_n v$ which is not in the image of I because $l' = -(n-1)$.

Term [4] gives $\langle i, n-1]\langle j, n-2]\sigma_{\bar{n}}\sigma_{n-1}\sigma_n v = \langle i, n-1]\sigma_{\bar{n}}\langle j, n]v$, so $k' = n+1$, $l' = i$, $u' = \langle j, n]v$. For this element to be in the image of I and of maximal length it is necessary that $u' = \langle j, n]v$ could be written as $\sigma_n u$, reduced form of a fc element, with $\langle i, n-2]\sigma_{\overline{n-1}}u$ satisfying the conditions in (18). This is impossible because $j < n$.

Finally term [3] gives $\langle i, n-1]\langle j, n-2]\sigma_n\sigma_{\bar{n}}\sigma_{n-1}v = \langle i, n-1]\sigma_{\bar{n}}\sigma_n\langle j, n-1]v$, so $k' = n+1$, $l' = i$, $u' = \sigma_n\langle j, n-1]v$. This element might be the image under I of $\langle i, n-2]\sigma_{\overline{n-1}}\langle j, n-1]v$. Here v belongs to $W^c(D_n)$ and we see directly from (18) and Definition 4.3 that if σ_{n-1} appears in v, then $\langle j, n-1]v$ cannot be reduced fc since $j < n-1$. On the other hand, if v belongs to $W^c(D_{n-1})$, conditions (18) for w imply conditions (18) for $\bar{w} = \langle i, n-2]\sigma_{\overline{n-1}}\langle j, n-1]v$, therefore in this case the element $pT_{I(\bar{w})}$, where $I(\bar{w}) = \langle i, n-1]\sigma_{\bar{n}}\sigma_n\langle j, n-1]v$ satisfies $l(I(\bar{w})) = l(I(w))$, appears in $R_n(h_w)$.

The lemma follows. □

Theorem 6.8. *The tower of affine Temperley-Lieb algebras*

$$TL\tilde{D}_3(q) \xrightarrow{P_3} TL\tilde{D}_4(q) \xrightarrow{P_4} \cdots TL\tilde{D}_n(q) \xrightarrow{P_n} TL\tilde{D}_{n+1}(q) \longrightarrow \cdots$$

is a tower of faithful arrows.

Proof. We need to show that P_n is an injective homomorphism of algebras. A basis for $TL\tilde{D}_n(q)$ is given by the elements h_w where w runs over $W^c(\tilde{D}_n)$. Assume there are non trivial dependence relations between the images of these basis elements. Pick one such relation, say

$$\sum_w \lambda_w P_n(h_w) = 0, \tag{25}$$

and let $m = \max\{L(w) \mid w \in W^c(\tilde{D}_n) \text{ and } \lambda_w \neq 0\}$.

If $m = 0$ or $m \geq 2$ the proof goes as in [4, Theorem 6.4], using Lemma 6.5 and Proposition 6.6, respectively.

If $m = 1$ we let:

$$\begin{aligned} X &= \{w \in W^c(\tilde{D}_n)/L(w) = 1 \text{ and } \lambda_w \neq 0\}, \\ l &= \max\{l(w)/w \in X\}, \\ X_0 &= \{w \in X/l(w) = l\}. \end{aligned}$$

We use Lemma 6.7 to develop the terms in (25) with $\lambda_w \neq 0$. We obtain terms T_z where z has affine length 0 or 1 and we want to set apart those T_z where z has affine length 1 and maximal Coxeter length. They are exactly the terms T_z where $L(z) = 1$ and $l(z) = l + 2$ in the development, using Lemma 6.7, of the sum

$$\sum_{w \in X_0} \lambda_w P_n(h_w).$$

Among those terms we have those T_z with $z \in \mathrm{Im}\ I$ and those T_z with $z \notin \mathrm{Im}\ I$, and the spaces they generate are in direct sum. So the following sum must be equal to 0:

$$\sum_{w \in X_0} \lambda_w \left(p^{\nu_w} T_{I(w)} + \delta_w \ p T_{I(\bar{w})} \right) \tag{26}$$

where we have set, with the notation in Lemma 6.7: $\delta_w = 1$ if $i < n$ and $v \in W^c(D_{n-1})$, and $\delta_w = 0$ (and $\bar{w} = w$) otherwise.

If this sum does not contain any w with $\delta_w = 1$, the injectivity of I shows directly that $\lambda_w = 0$ for $w \in X_0$, a contradiction. Otherwise we recall from (24) that the image by I of terms of affine length 1 is the disjoint union of two subsets, one is the image A of those terms with $i = n$, the other the image B of those terms with $i < n$.

We notice that if $\delta_w = 1$, then $I(w) \in B$ and $I(\bar{w}) \in A$. The linear combination of the terms T_x with $x \in B$ in (26) has to be zero, and it is equal to

$$\sum_{w \in X_0, I(w) \in B} \lambda_w T_{I(w)},$$

so the coefficients λ_w, for $w \in X_0$ and $\delta_w = 1$, must be 0, again a contradiction. $\square$

REFERENCES

[1] S. Al Harbat. On the affine braid group, affine Temperley-Lieb algebra and Markov trace. PH.D Thesis, Université Paris-Diderot-Paris 7, 2013.

[2] S. Al Harbat. Markov trace on a tower of affine Temperley-Lieb algebras of type $\tilde{A}$. J. Knot Theory Ramifications 24 (2015) 1–28. DOI : 10.1142/S0218216515500492

[3] S. Al Harbat. Tower of fully commutative elements of type $\tilde{A}$ and applications. J. Algebra 465 (2016), 111–136.

[4] S. Al Harbat. On the fully commutative elements of type $\tilde{C}$ and faithfulness of related towers. J. Algebraic Combin. 45 (2017), no. 3, 803–824.

[5] R. Biagioli, F. Jouhet, P. Nadeau. Fully commutative elements in finite and affine Coxeter groups. Monatsh. Math. 178 (2015), no. 1, 1–37.

[6] N. Bourbaki. Groupes et algèbres de Lie, chapitres 4, 5, 6. Masson, 1981.

[7] V.V. Deodhar. A note on subgroups generated by reflections in Coxeter groups. Arch. Math., Vol. 53 (1989), 543–546.

[8] J.J. Graham. Modular representations of Hecke algebras and related algebras. Ph.D. Thesis, University of Sydney, 1995.

[9] R.M. Green. Leading coefficients of Kazhdan-Lusztig polynomials and fully commutative elements. J. Algebraic Combin. 30 (2009), no. 2, 165–171.

[10] C.R.H. Hanusa, B.C. Jones. The enumeration of fully commutative affine permutations. European J. Combin. 31 (2010), no. 5,134–1359.

[11] G. Lusztig. Hecke algebras with unequal parameters. CRM Monograph Series, 18. American Mathematical Society, Providence, RI, 2003.

[12] T.J. McLarnan, G.S. Warrington. Counterexamples to the 0-1 conjecture. Represent. Theory 7 (2003), 181–195.

[13] J. R. Stembridge. Some combinatorial aspects of reduced words in finite Coxeter groups. Transactions A.M.S. 349 (4) 1997, 1285–1332.

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