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Groupe de tresses affine, algèbre de Temperley-Lieb affine et trace de Markov

Affine braid group, affine Temperley-Lieb algebra and Markov trace

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Abstract

In this thesis we define a tower of affine Temperley-Lieb algebras of Type $\tilde{A}$ on which we define a Markov trace and we show that there a unique such trace.

In order to do so, we work on four levels of type $\tilde{A}$: affine braid groups, affine Coxeter groups, affine Hecke algebras and affine Temperley-Lieb algebras.

On the braid level, we show that $\tilde{A}$-type affine braid group with $n+1$ generators $B(\tilde{A}_n)$ surjects onto A-type affine braid group with n generators $B(A_n)$, we prove that this surjection comes from a quotient on a certain subgroup of $B(\tilde{A}_n)$ and we define a closure of an element of this group which is to be called an affine link.

On the Coxeter level, we study the $\tilde{A}$-type affine Coxeter group with $n+1$ generators $W(\tilde{A}_n)$, we give a full set of representatives of $W(\tilde{A_{n-1}})/W(\tilde{A}_n)$ and $W(\tilde{A_{n-1}})/W(\tilde{A}_n)/W(\tilde{A_{n-1}})$. Then we classify fully commutative elements in $W(\tilde{A}_n)$ and we give a normal form for such elements.

On the Hecke level, we define a tower of $\tilde{A}$-type affine Hecke algebras, we show that this tower is a tower of inclusions, and we show that this tower "surjects onto" the tower of A-type Hecke algebras.

On the Temperley-Lieb level, we define a tower of $\tilde{A}$-type affine Temperley-Lieb algebras namely $(\widehat{TL}_{n+1}(q))_{0\geq n}$, we define a Markov trace as a collection of traces $(\hat{\tau}_{n+1}(q))_{0\geq n}$ in its most general form (compatible with affine links). We get the existence of such trace by showing that the mentioned tower "surjects onto" the tower of A-type Temperley-Lieb algebras, and finally we show that this trace is unique by making use of the normal form of the fully commutative elements in $W(\tilde{A}_n)$.

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Chapter

1 Introduction

In these days the angel of topology and the devil of abstract algebra fight for the soul of each individual mathematical domain, said H. Weyl, about 60 year ago. Nevertheless, some times the angel of topology itself was seduced by the devil of algebra! The well-known work of Jones is a remarkable example of such a case. The problem was to find an invariant for links in $\mathbb{R}^3$, Alexander built a bridge between braids and links by showing every link is a closure of some braid, then Markov conjectured certain conditions for two braids, so that they give the same link. Alexander defined the polynomials Δ_t, named after him, to be an invariant of links. We had to wait about fifty years, so that, the young Jones defines his polynomials, which were very easy to be computed, compared with Alexander polynomials. Moreover, Jones' solution contained Alexander's, since, in Jones' work, Alexander polynomials can be viewed as a special case. Although Jones polynomials do not distinguished all the oriented links, it is still the most powerful invariant, until nowadays. Such topological problem finds an answer using pure algebraical tools, precisely group theory, the angel could not help being seduced, the devil is as guilty as charged!

The Markov-Ocneanu-Jones trace is a collection of traces $(\tau_{n+1})_{0\leq n}$, defined on the tower of Hecke algebras:

$$H_0(q) \ \subset H_1(q) \ \ ... \subset H_{n-1}(q) \quad \subset H_n(q) \ \ ...$$

Under the conditions of Markov we have many traces, they can be parametrized by some z in the ground ring. Nevertheless, Jones had started to construct such trace, on a tower of quotients of Hecke algebras: the tower of Temperley-Lieb algebras:

$$TL_0(q) \ \subset TL_1(q) \ \ ... \subset TL_{n-1}(q) \quad \subset TL_n(q) \ \ ...$$

For a certain choice of z, this trace is factorized through the T-L tower, it is unique. Its values on the elements of T-L algebras (which are surjective images of the braid group algebra) are the well-known Jones polynomials.

My recent work, in my thesis, is an attempt to answer Digne's question: what does happen in the affine case? I could find an answer up to the affine T-L level. I get to make it clear that my work was not about defining an invariant for affine knots (even the results gives such invariant). It is really about defining an affine Markov trace and studying this trace, i did not treat Markov trace as a tool, towards knot theory. It is to be viewed as an independent "algebraic being". The answer is: there exist a unique affine Markov trace on the tower of affine T-L algebras, coming from the unique Markov trace on the tower of T-L algebras (After showing that $\widehat{TL}_{n+1}(q)$ surjects onto $TL_n(q)$). Moreover, the restriction of this affine trace to $(TL_n(q))_{0\geq n}$, viewed as sub-algebras of $(\widehat{TL}_{n+1}(q))_{0\geq n}$ is the Markov-Ocneanu-Jones trace. Since the answer is so, The problem of classifying all the affine Markov traces on $\widehat{H}_{n+1}(q)_{0\geq n}$ is naturally the second step, I am particularly concerned in such this problem, although, my tries leads to have a guess that we do have more than "z" traces, unlike the finite dimension case.

Here arise a question, after considering the paper of Green about the type E_n [Gre09b] where we see a unique Markov trace, it seems that the uniqueness of the Markov trace has a friendly relation with the equality of the parameters! in other terms: we almost have a unique trace whenever the T-L algebra is associated to a Coxeter group with conjugate generators, no matter finite or affine, " almost", because the type D_n has maybe a singularity, which is one of the point i am attending to focus on in the near future, the Hecke level was treated in [GL97] by Geck and Lambropoulou.

Since we focus on the tower

$$\widehat{TL}_1(q) \xrightarrow{F_1} \widehat{TL}_2(q) \xrightarrow{F_2} \widehat{TL}_3(q) \longrightarrow \quad ... \ \widehat{TL}_n(q) \xrightarrow{F_n} \widehat{TL}_{n+1}(q) \longrightarrow ...,$$

we see that there will be a strong presence of the fully commutative elements in the related affine Coxeter group $W(\tilde{A}_n)$. In our way we will give a normal form of those elements.

The skeleton of the proof is pretty long. We start by presenting a fully commutative element w in its normal form, which we found in 3.4.4. Then we left this element to $\bar{w}$ in $B(\tilde{A}_n)$ which has the same reduced expression. Then we obtain a kind of canonical form of this $\bar{w}$ in corollary 3.5.2. Then we send $\bar{w}$ to its image via the surjection from $K(B(\tilde{A}_n)]$ onto $\widehat{TL}_{n+1}(q)$. Now we consider the fact that $g_{\sigma_n \dots \sigma_1 a_{n+1}}$ in $\widehat{TL}_{n+1}(q)$ acts on $F_n(\widehat{TL}_n(q))$ as a Dynkin automorphism, together with the fact that $\widehat{TL}_{n+1}(q)$ surjects onto $TL_n(q)$ we prove the existence of an affine Markov trace, then we prove the uniqueness of this trace.

This work is divided into five chapters;

In **chapter 2** we give some brief definitions of the $\tilde{A}$-type affine braid group and we view it as a subgroup of the B-type braid group. We propose a "parabolic-like" presentation of $W(\tilde{A}_n)$. Then in theorem 2.4.7 we show our first result: viewing $B(A_n)$ (the A-type braid group) as a quotient of $B(\tilde{A}_n)$. Then we detect the kernel of this surjection using Schreier's theorem. Then we show that the surjection above and other arrows are compatible with towers of braid groups. Finally, we give a definition of affine knots, their closures, and we prove that the normal Markov conditions are necessary and sufficient to get the same affine closure of any two affine braids.

In **chapter 3** we see first, why the new presentation is "parabolic-like", we see as well that the well-know semi direct presentation of $W(\tilde{A}_n)$ is compatible with the semi direct product of $B(\tilde{A}_n)$ which we have seen in chapter 1. Our first result is given in proposition 3.3.5 and corollary 3.3.7 about the left and double classes of $W(\tilde{A_{n-1}})$ in $W(\tilde{A}_n)$. The main result of this chapter is theorem 3.4.4, where we can found a normal form of the elements of $W^c(\tilde{A}_n)$.

In **chapter 4** we give some definitions and known results, then we define Markov trace. Our main result here is proposition 4.3.3, where we see that the tower of affine Hecke algebras is a tower of inclusions. Then we give the "Hecke" version of theorem 2.4.7.

In **chapter 5** we give a summary about Markov trace, and some base changes by explaining the simplifications of J. Michel. Then we give the "TL" version of theorem 2.4.7. Then we consider the tower of affine T-L algebras. Then we show for the compatibility between the towers of algebras and the surjection we have found from the affine level to the finite-dimensional level.

In **chapter 6** we introduce the concept of Markov elements, then we proof a general result, theorem 6.1.1, about a set of elements which are sufficient to define any trace uniquely. In proposition 6.1.10 we classify all traces on $\widehat{TL}_2(q)$ which have the same value on the two generators, we do almost the same with $\widehat{TL}_3(q)$ in proposition 6.1.12. Then we give the definition of an affine Markov trace, we show in proposition 6.2.4 and corollary 6.2.5 that there exist an affine Markov trace coming from the surjection of $\widehat{TL}_{n+1}(q)$ onto $TL_n(q)$. Then we prove the uniqueness passing by corollary 6.2.7, and finally we announce the general result in theorem 6.2.8.

We point out that the star (*) in any: section, subsection, theorem or proposition; means that there is a related explication in the section called "Bibliographical remarks and problems" at the end of every chapter.

و نمْ يا حبيبي, عليكَ ضفائر شَعري .. عليك السّلام. (م. درويش)

Close your eyes, peace be with you my love,

may my braids be upon you. (M.D)

Chapter 2

Artin-Tits braid groups

In some universe of groups (so-called Category of groups), Braid groups and Artin-Tits groups have the same "DNA" but they are not the same soul, in other words they are isomorphic but not in a unique way. Since we have many choices of isomorphisms between them, these two groups are not the same Object in this category. In what follows we describe briefly the definitions making our choices once and for all. We are interested in the pure algebraic presentation (generators and relations) which is the "Artin" way together with the geometric presentation (strings and familiar "braids") which is more likely the braids way.

2.1 Artin groups and Braid groups *

Let S be a finite set.

Definition 2.1.1. *A Coxeter matrix over S is a square matrix $M = (m_{st})_{s,t \in S}$ such that*

- $m_{ss} = 1$,
- $m_{st} = m_{ts}$ *for any* s, t *in* S,
- m_{st} *belongs to* $\{2, 3, 4 ... \infty\}$.

We present a Coxeter matrix by its Dynkin graph $\Gamma = \Gamma(M)$, which is a graph given by vertices and edges. Γ has S as a set of vertices, and for any non-equal two vertices s, t in S we have

- s, t are joined by an edge if $m_{s,t} = 3$,
- s, t are joined by a doubled edge if $m_{s,t} = 4$,
- s, t are joined by an edge labeled by $m_{s,t}$ if $m_{s,t} > 4$.

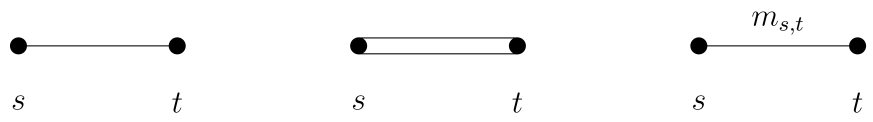


Figure 2.1: Edges

Definition 2.1.2. *Let B_S be the set $\{\sigma_s;\ s \in S\}$. We call the pair (B, S) an Artin system of type Γ, where $B = B_\Gamma$ is the group given by generators and relations as follows: S is the set of generators with relations $prod(\sigma_s, \sigma_t : m_{st}) = prod(\sigma_t\sigma_s : m_{st})$, for any non-equal s, t in S with $m_{s,t} \neq \infty$.*

We call B the Artin group of type Γ.

In our work we treat many kinds of Artin groups in which $m_{s,t} \leq 4$. The type of relations appears in the definition is called "braid relations".

Let Γ be a Dynkin graph. Let (B, S) be the related Artin system. Take V to be the real vector space with $\mathbb{R}$-basis $\{e_s; s \in S\}$ which plays the role of the set of simple roots. The root system gives rise to simple reflections hence to a reflection group generated by those simple reflections, say W_S. By the natural linear representation of W_S we can realize it as a subgroup in $GL(V)$ (the group of endomorphisms of V). Let R be the set of reflections of W_S (the set of conjugates of simple reflections). Take r in R, since it is a reflection it fixes a hyper-plane in V, say H_r. In fact W_S acts freely on the complement of $\cup_{r\in R} H_r$ in V. By extending the action of W_S up to $V_\mathbb{C} = \mathbb{C} \otimes_\mathbb{R} V$ we see that W_S acts freely on the complement of $\cup_{r\in R}\mathbb{C} \otimes_\mathbb{R} H_r$ in $V_\mathbb{C}$. We call this complement M_Γ. We set $N_\Gamma = M_\Gamma / W_S$.

Remark 2.1.3. *Before stating the theorem, we have to notice that the above argument is valid in the case of finite W_S. For when the group W_S is infinite we have to replace V by $U \subset V$ (the so-called Tits cone), and to replace as well $V_\mathbb{C}$ by $(U + iV) \subset V_\mathbb{C}$. M_Γ is to be $(U + iV) - \cup_{r\in R}\mathbb{C} \otimes_\mathbb{R} H_r$. The action of W_S on M_Γ is free, and as above N_Γ is M_Γ modulo the action of W_S.*

Definition 2.1.4. *The braid group of Γ-type is $\pi_1(N_\Gamma)$, the fundamental group of the space N_Γ.*

Theorem 2.1.5. *(Brieskorn-Van der Lek). $\pi_1(N_\Gamma) \simeq B$.*

Briefly: the Γ-type Artin group is given by a presentation, while the Γ-type braid group is a fundamental group. Although the Brieskorn-Van der Lek isomorphism is not canonical, we will not make the distinction in this work. We will call each of these groups a Γ-type braid group.

2.2 A-type braid groups *

An A-type braid group with n generators is historically the first braid group. We give its presentation by generators and relations, then a geometrical one (by means of braids with $n+1$ strands). Many interesting basic facts show the reasons for which it has such respectable position in the group theory, in addition to many other branches of mathematics for example it has a faithful representation in $Aut(F_{n+1})$, the group of automorphisms of the free group with n generators. The strong relations with link theory (here comes the well known Alexander theorem). In addition of its Garsiditude, in fact, it is the first group to be called a Garside group.

2.2.1 Presentations

Let $n \geq 1$ be an integer.

Definition 2.2.1. *The A-type braid group $B(A_n)$ with n generators is the group presented by a generator set $S = \{\sigma_1, \sigma_2, .., \sigma_n\}$ and the relations*

- $\sigma_i\sigma_j = \sigma_j\sigma_i$ *where* $1 \leq i, j \leq n$ *and* $|i - j| \geq 2$,
- $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ *where* $1 \leq i \leq n-1$.

Thus the related Dynkin diagram is

Figure 2.2: Γ_A

Let $P_n, .., P_{n+1}$ be distinct points in the plane $\mathbb{R}^2$. We define a rough braid on $n+1$ strands to be an n-tuple $\beta = (b_1, .., b_{n+1})$, where b_k is a path $b_k : [0,1] \to \mathbb{R}$ such that

- For any k in $\{1, .., n+1\}$ we have $b_k(0) = P_k$,
- For any k in $\{1, .., n+1\}$ there exists a permutation $x = \theta(\beta) \in Sym_{n+1}$ such that $b_k(1) = P_{x(k)}$,

- For any non-equal k and l in $\{1,..,n+1\}$, for all $t \in [0,1]$ we have $b_k(t) \neq b_l(t)$.

By definition: two rough braids α and β are homotopic if there exists a continuous family of rough braids $\{\gamma_s\}_{s\in[0,1]}$ such that $\gamma_0 = \alpha$ and $\gamma_1 = \beta$. This is an equivalence relation.

Definition 2.2.2. *A braid on $n+1$ is a homotopy classes of rough braids on $n+1$ strands.*

The well known geometric interpretation of the elements of $B(A_n)$ viewed as braids in the space is the following

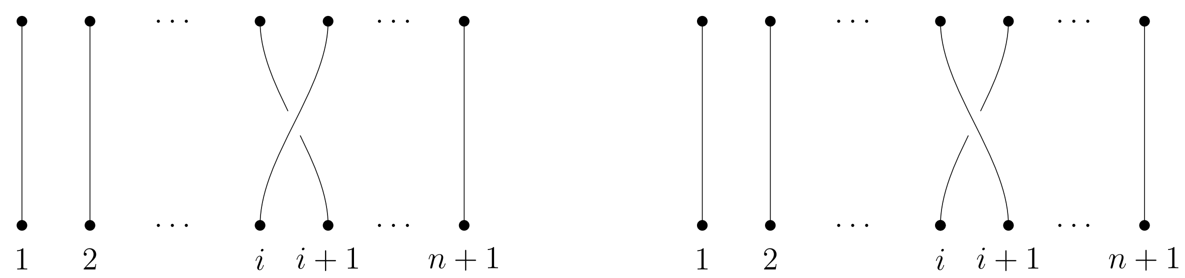


Figure 2.3: σ_i & σ_i^{-1}

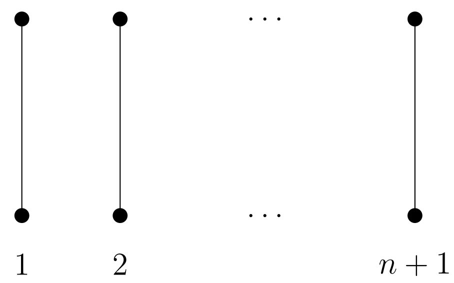


Figure 2.4: Id

We compose two braids in the way that one would expect, that is for any two braids X, Y the composed braid XY is the braid obtained by putting X at the top and Y at the bottom, welding the bottom end points of X with the upper ones of Y (the i-th with the i-th, $1 \leq i \leq n+1$) as follows:

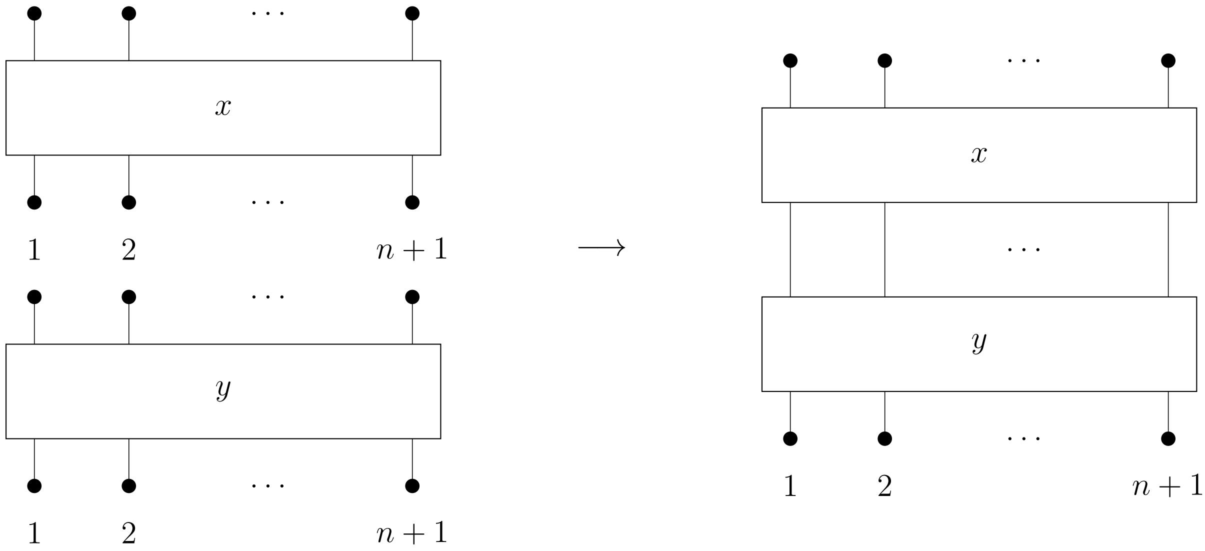


Figure 2.5: $X, Y \to XY$

The natural embedding

$$\begin{aligned} x_{n-1} : B(A_{n-1}) &\longrightarrow B(A_n) \\ \sigma_i &\longmapsto \sigma_i \text{ for } 1 \leq i \leq n-1, \end{aligned}$$

can be realized geometrically by adding the $(n+1)$-th stand

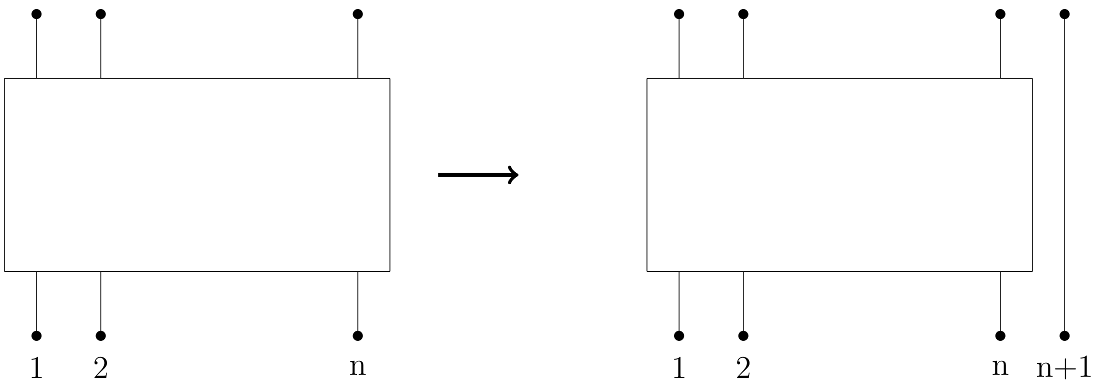


Figure 2.6: x_{n-1}

2.2.2 A faithful representation

We can realize $B(A_n)$ as a subgroup of the group of automorphisms of the free group with $n+1$ generators via the "Artin representation", which will be briefly defined in what follows: Let F_{n+1} be the free group with $n+1$ generators $x_1, .., x_{n+1}$. Let $Aut(F_{n+1})$ be the group of automorphisms of F_{n+1}. For $1 \leq k \leq n$, we define t_k in $Aut(F_{n+1})$ as follows for $(i \neq k, k+1)$:

$$\begin{aligned} t_k : F_n &\longrightarrow F_n \\ x_i &\longmapsto x_i, \\ x_k &\longmapsto x_n^{-1} x_{k+1} x_k, \\ x_{k+1} &\longmapsto x_k. \end{aligned}$$

It is easy to show that the map $\rho : B(A_n) \longrightarrow Aut(F_{n+1})$, which sends σ_k to t_k, defines a representation of $B(A_n)$ in $Aut(F_{n+1})$ called the Artin representation.

Theorem 2.2.3. *(Artin) The representation ρ is faithful.*

2.3 B-type braid groups *

The B-type braid group with $n+1$ generators $B(B_{n+1})$ plays a role in the theory of low dimension topological spaces, in addition to the fact that it is very useful in investigating the structure of the affine braid group which is the center of interest of this work in general.

2.3.1 Presentations

Definition 2.3.1. *The B-type braid group with $n+1$ generators $B(B_{n+1})$ is the group presented by a generators set $\{\sigma_1, \sigma_2..\sigma_n, t\}$ and the relations*

- $\sigma_i \sigma_j = \sigma_j \sigma_i$ *where* $1 \leq i, j \leq n$ *when* $|i-j| \geq 2$,
- $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ *when* $1 \leq i \leq n$,
- $\sigma_i t = t \sigma_i$ *when* $2 \leq i \leq n$,
- $\sigma_1 t \sigma_1 t = t \sigma_1 t \sigma_1$.

The related Dynkin diagram is

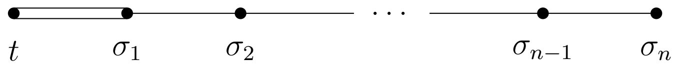


Figure 2.7: $\Gamma_{B_{n+1}}$

Set $\phi_{n+1} = t\sigma_1..\sigma_n$. Set $a_{n+1} = \phi_{n+1}\sigma_n\phi_{n+1}^{-1}$. We can see directly that $\phi_{n+1}\sigma_i\phi_{n+1}^{-1} = \sigma_{i+1}$ for all $1 \leq i \leq n-1$, with $\phi_{n+1}\sigma_n\phi_{n+1}^{-1} = a_{n+1}$. What is more, we have the following presentation:

Proposition 2.3.2. *$B(B_{n+1})$ is presented by the set of generators $S' = \{\sigma_1, \sigma_2..\sigma_n, a_{n+1}, \phi_{n+1}\}$ together with the relations*

- $\sigma_i\sigma_j = \sigma_j\sigma_i$ *where* $1 \leq i, j \leq n$ *when* $|i-j| \geq 2$,
- $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ *when* $1 \leq i \leq n-1$,
- $\sigma_i a_{n+1} = a_{n+1}\sigma_i$ *when* $2 \leq i \leq n-1$,
- $\sigma_1 a_{n+1}\sigma_1 = a_{n+1}\sigma_1 a_{n+1}$,
- $\sigma_n a_{n+1}\sigma_n = a_{n+1}\sigma_n a_{n+1}$,
- $\phi_{n+1}\sigma_i\phi_{n+1}^{-1} = \sigma_{i+1}$ *when* $1 \leq i \leq n-1$,
- $\phi_{n+1}\sigma_n\phi_{n+1}^{-1} = a_{n+1}$,
- $\phi_{n+1}a_{n+1}\phi_{n+1}^{-1} = \sigma_1$.

Notice that the relations involving ϕ_{n+1} are not braid relations, i.e., $(B(B_{n+1}), S')$ cannot be viewed as an Artin system.

Considering those relations we see directly that ϕ_{n+1} is acting by automorphisms on the normal subgroup of $B(B_{n+1})$ generated by $\{\sigma_1, \sigma_2..\sigma_n, a_{n+1}\}$. More, this set $\{\sigma_1, \sigma_2..\sigma_n, a_{n+1}\}$ with the first five systems of relations form a presentation by generators and relations of this subgroup which will be the subject of the next section. The element ϕ_{n+1} generates a free subgroup of rank 1, which we denote by Φ_{n+1}. In other terms:

$$B(B_{n+1}) = \langle \sigma_1, \sigma_2..\sigma_n, a_{n+1} \rangle_{B(B_{n+1})} \rtimes \Phi_{n+1}.$$

The geometric presentation of $B(B_{n+1})$ as braids is given in a very similar way of that of $B(A_n)$ but in $n+2$ strands, where the first strand remains point-wise fixed. The generators are presented as follows:

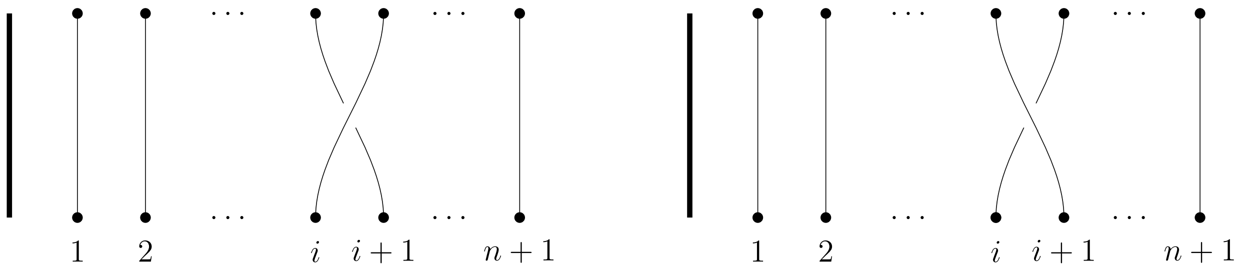


Figure 2.8: σ_i & σ_i^{-1}

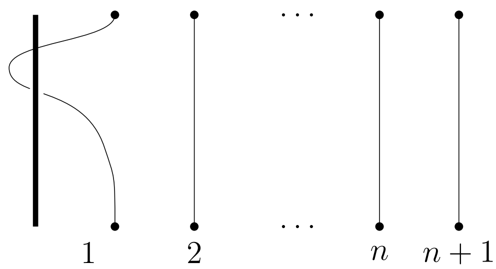


Figure 2.9: t

2.4 Affine braids: the group $B(\tilde{A}_n)$

The $\tilde{A}$-type affine braid group in $n+1$ generators is the braid group under question in this work. Geometrically, one can see several presentations in the literature, among which we choose one which is compatible with our viewpoint on this group (as a base point for a better understanding of a special kind of links in the space). We show the strong connection between this group and the two groups mentioned above. The arrows in this section are well known, except for the surjection of $B(\tilde{A}_n)$ onto $B(A_n)$ which allows us to see $B(\tilde{A}_n)$ as a semi-direct product of $B(A_n)$ with a "huge" group, an infinitely generated free group (the semi-direct does not seem of a great use). In the other hand this surjection (and others induced by it) plays an essential role in the solution of the main question which this work is attempting to answer. While concerning $B(\tilde{A}_n)$ presentations, we define a new presentation (not far from the old one) called the parabolic-like presentation.

2.4.1 Presentations

Definition 2.4.1. *The $\tilde{A}$-type braid group with $n+1$ generators $B(\tilde{A}_n)$ is the group presented by a set of generators $\{\sigma_1, \sigma_2..\sigma_n, a_{n+1},\}$ together with the following defining relations*

- $\sigma_i\sigma_j = \sigma_j\sigma_i$ *where* $1 \le i, j \le n$ *when* $|i-j| \ge 2$,
- $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ *when* $1 \le i \le n-1$,
- $\sigma_i a_{n+1} = a_{n+1}\sigma_i$ *when* $2 \le i \le n-1$,
- $\sigma_1 a_{n+1}\sigma_1 = a_{n+1}\sigma_1 a_{n+1}$,
- $\sigma_n a_{n+1}\sigma_n = a_{n+1}\sigma_n a_{n+1}$.

Remark 2.4.2. *In the literature a_{n+1} in this definition is often called σ_{n+1}, but since we are interested in viewing $B(\tilde{A_{n-1}})$ as a subgroup of $B(\tilde{A}_n)$ for $2 \le n$, there would be a confusion between a_n and σ_n when they are seen as elements in $B(\tilde{A}_n)$. Thus we consider the group $B(A_n)$ generated by $\{\sigma_1, \sigma_2..\sigma_n\}$, then we "affinize" it by adding a_{n+1}.*

Remark 2.4.3. *It is not a coincidence that the generators here have the same symbols as the generators of $B(B_{n+1})$ in the second presentation of $B(B_{n+1})$. there is an obvious homomorphism $B(\tilde{A}_n) \longrightarrow B(B_{n+1})$, saying that this homomorphism is injective is equivalent to saying that in $B(B_{n+1})$ the subgroup generated by the elements $\sigma_1, \sigma_2..\sigma_n, a_{n+1}$ is presented by generators and relations in the following way: it has for generating set $\{\sigma_1, \sigma_2..\sigma_n, a_{n+1},\}$ together with the relations*

- $\sigma_i\sigma_j = \sigma_j\sigma_i$ *where* $1 \le i, j \le n$ *when* $|i-j| \ge 2$,
- $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ *when* $1 \le i \le n-1$,
- $\sigma_i a_{n+1} = a_{n+1}\sigma_i$ *when* $2 \le i \le n-1$,
- $\sigma_1 a_{n+1}\sigma_1 = a_{n+1}\sigma_1 a_{n+1}$,
- $\sigma_n a_{n+1}\sigma_n = a_{n+1}\sigma_n a_{n+1}$.

This is true, since ϕ_{n+1} acts on this very group by automorphisms, thus the relations between its generators do not add any other relation than the braid respecting length relations already existing in the definition.

Proposition 2.4.4. *Let x be in $B(B_{n+1})$. Suppose that x is expressed as a word in the generators: $\sigma_1, \sigma_2..\sigma_n, t$. Then*

$$x \in B(\tilde{A_n}) \Longleftrightarrow \text{the sum of the exponents of } t \text{ in } x \text{ is zero.}$$

Proof. Suppose that $x = u_1 t^{b_1} u_2 t^{b_2} u_3..u_m t^{b_m} u_{m+1}$, where b_i is integer and u_i is in $B(A_n)$ for all i.

We have $\phi_{n+1} = t\sigma_1..\sigma_{n-1}\sigma_n$ by definition . Set $z^{-1} = \sigma_1..\sigma_{n-1}\sigma_n$, that gives $t = \phi_{n+1}z$. We denote the action of ϕ_{n+1}^r on an element e in $B(\tilde{A_n})$ by $[e]^r$, for any integer r. For example $\phi_{n+1}z = [z]^1\phi_{n+1}$.

Now $t^{b_i} = \underbrace{\phi_{n+1}z\phi_{n+1}z..\phi_{n+1}z}_{b_i \text{ times}}$, which is equal to $\phi_{n+1}^{b_i}\prod\limits_{j=0}^{j=b_i}[z]^{j-b_i}$.

Set $Z_{b_i} = \prod\limits_{j=0}^{j=b_i}[z]^{j-b_i}$, which is in $B(\tilde{A_n})$. Thus:

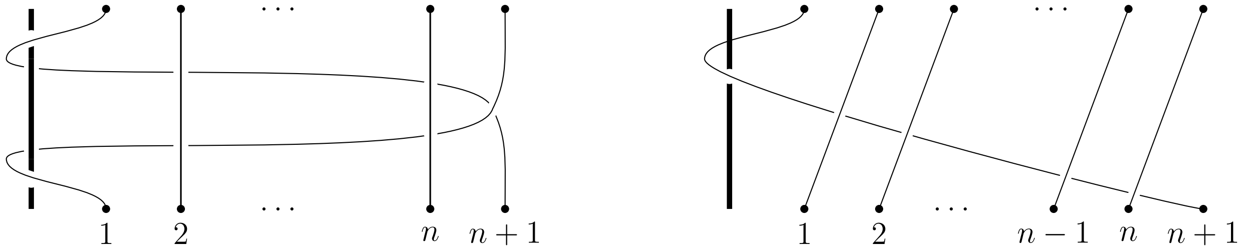
$x = u_1\phi^{b_1}Z_{b_1}u_2\phi^{b_2}Z_{b_2}u_3..u_m\phi^{b_m}Z_{b_m}u_{m+1}$. By pushing the "$\phi^{b_i}$"s to the right (acting on the "u_i"s as well as on the "Z_{b_1}"s) we get the following expression of x:

$x = \lambda\phi^{\left(\sum\limits_{i=1}^{i=m} b_i\right)}$ where $\lambda \in \tilde{A_n}$. This is the unique decomposition of x in the semi-direct product $\langle\sigma_1, \sigma_2..\sigma_n, a_{n+1}\rangle_{B(B_{n+1})} \rtimes \Phi_{n+1}$.

Now x is in $B(\tilde{A_n})$ if and only if $\sum\limits_{i=1}^{i=m} b^i = 0$. The proposition follows.

□

Inspired by the injection $B(\tilde{A_n}) \hookrightarrow B(B_{n+1})$, we explain the geometric presentation we choose in this work. Actually, affine braids with $n+1$ generators could be viewed for example as cylindrical braids within a cylinder labeled by $n+1$ points on each of its circles, the strings of the braids are not allowed to make perfect rounds. We choose to view the affine braids as B-type braids: using the proposition above, the affine braids are B-type braid in which the number of positive rounds equals the number of negative rounds (around the fixed strand). The "σ"s are as presented above, while a_{n+1} and ϕ_{n+1} the following braids:



(a_{n+1}) (ψ_{n+1})

Figure 2.10:

Notice that ϕ_{n+1} is not an affine braid.

Proposition 2.4.5. *The following group homomorphism is injective*

$$F_n : B(\tilde{A_{n-1}}) \longrightarrow B(\tilde{A}_n)$$
$$\sigma_i \longmapsto \sigma_i \quad \textit{for } 1 \leq i \leq n-1$$
$$a_n \longmapsto \sigma_n a_{n+1} \sigma_n^{-1}$$

Proof. See 2.4.4 F_n is a restriction of the injection y_n to $B(\tilde{A}_n)$. $\square$

We give now a new presentation of $B(\tilde{A}_n)$, in which the defining relations are positive, and where F_n is obtained by simply adding one generator to those of $B(\tilde{A_{n-1}})$.

By definition $B(\tilde{A}_n)$ has $\{\sigma_1, \sigma_2 .. \sigma_n, a_{n+1}\}$ as a set of generators together with the following defining relations:

(1') $\sigma_i \sigma_j = \sigma_j \sigma_i$ where $1 \leq i, j \leq n$ when $|i - j| \geq 2$,

(2') $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ when $1 \leq i \leq n-1$,

(3') $\sigma_i a_{n+1} = a_{n+1} \sigma_i$ when $2 \leq i \leq n-1$,

(4') $\sigma_1 a_{n+1} \sigma_1 = a_{n+1} \sigma_1 a_{n+1}$,

(5') $\sigma_n a_{n+1} \sigma_n = a_{n+1} \sigma_n a_{n+1}$ for $n \geq 2$.

This presentation is to be called the formal presentation from now on. For the moment n is to be greater than or equal to 3. $B(\tilde{A}_2)$ is generated by σ_1, σ_2 and a_3, we see that a_{n+1} can be seen as follows:

$$a_{n+1} = \sigma_n^{-1}..\sigma_3^{-1}a_3\sigma_3..\sigma_n.$$

Our aim is to show that $B(\tilde{A}_n)$ can be generated by $S' = \{\sigma_1, \sigma_2..\sigma_n, a_3\}$ with defining relations:

(1) $\sigma_i\sigma_j = \sigma_j\sigma_i$ where $1 \leq i, j \leq n$ when $|i - j| \geq 2$,

(2) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ when $1 \leq i \leq n-1$,

(3) $\sigma_1 a_3\sigma_1 = a_3\sigma_1 a_3$,

(4) $\sigma_3 a_3\sigma_3 = a_3\sigma_3 a_3$,

(5) $\sigma_i a_3 = a_3\sigma_i$ when $4 \leq i \leq n$,

(6) $\sigma_3\sigma_2 a_3\sigma_3 = \sigma_2 a_3\sigma_3\sigma_2$.

First we show that the formal presentation gives the new one, now $\{\sigma_1, \sigma_2..\sigma_n, a_3\}$ generates $B(\tilde{A}_n)$ indeed, since

$$a_{n+1} = \sigma_n^{-1}..\sigma_3^{-1}a_3\sigma_3..\sigma_n, \qquad \text{which gives} \qquad a_3 = \sigma_3..\sigma_n a_{n+1}\sigma_n^{-1}..\sigma_3^{-1}.$$

We see that (3) follows directly from the fact that a_3 and a_{n+1} are conjugate, while (5) could be seen to be valid geometrically or by a direct computation:

$$\text{for } 4 \leq i \leq n, \text{ we have : } \sigma_i a_3 = \sigma_i\sigma_3..\sigma_n a_{n+1}\sigma_n^{-1}..\sigma_3^{-1} = \sigma_3..\sigma_n\sigma_{i-1}a_{n+1}\sigma_n^{-1}..\sigma_3^{-1}$$

$$= \sigma_3..\sigma_n a_{n+1}\sigma_{i-1}\sigma_n^{-1}..\sigma_3^{-1} = \sigma_3..\sigma_n a_{n+1}\sigma_n^{-1}..\sigma_3^{-1}\sigma_i$$

$$= a_3\sigma_i.$$

Now we treat (4):

$$\sigma_3 a_3\sigma_3 = \sigma_3^2..\sigma_n a_{n+1}\sigma_n^{-1}..\sigma_4^{-1}.$$

$$\begin{aligned}
\text{Thus, } a_3\sigma_3 a_3 &= \sigma_3..\sigma_n a_{n+1} \underbrace{\sigma_n^{-1}..\sigma_4^{-1}\sigma_3\sigma_4..\sigma_n} a_{n+1}\sigma_n^{-1}..\sigma_3^{-1} \\
&= \sigma_3..\sigma_n a_{n+1}\sigma_3..\sigma_{n-1}\sigma_n\sigma_{n-1}^{-1}..\sigma_3^{-1}a_{n+1}\sigma_n^{-1}..\sigma_3^{-1} \\
&= \sigma_3..\sigma_n\sigma_3..\sigma_{n-1}a_{n+1}\sigma_n a_{n+1}\sigma_{n-1}^{-1}..\sigma_3^{-1}\sigma_n^{-1}..\sigma_3^{-1} \\
&= \sigma_3..\sigma_n\sigma_3..\sigma_{n-1}\sigma_n a_{n+1}\sigma_n\sigma_{n-1}^{-1}..\sigma_3^{-1}\sigma_n^{-1}..\sigma_3^{-1} \\
&= \sigma_3..\sigma_n\sigma_3..\sigma_{n-1}\sigma_n a_{n+1}\sigma_n\sigma_n^{-1}..\sigma_3^{-1}\sigma_n^{-1}..\sigma_4^{-1} \\
&= \sigma_3..\sigma_n \underbrace{\sigma_3..\sigma_{n-1}\sigma_n\sigma_{n-1}^{-1}..\sigma_3^{-1}} a_{n+1}\sigma_n^{-1}..\sigma_4^{-1} \\
&= \sigma_3..\sigma_n\sigma_n^{-1}..\sigma_4^{-1}\sigma_3\sigma_4..\sigma_n a_{n+1}\sigma_n^{-1}..\sigma_4^{-1} \\
&= \sigma_3^2\sigma_4..\sigma_n a_{n+1}\sigma_n^{-1}..\sigma_4^{-1} \\
&= \sigma_3 a_3\sigma_3.
\end{aligned}$$

We see that (6) is equivalent to

$$\sigma_2\sigma_3^{-1}a_3\sigma_3 = \sigma_3^{-1}a_3\sigma_3\sigma_2 \qquad ...(6'').$$

But $\sigma_3^{-1}a_3\sigma_3 = \sigma_4..\sigma_n a_{n+1}\sigma_n^{-1}..\sigma_4^{-1}$, and (6) follows.

Now we show that the new presentation gives the formal one.

We are reduced to show that the new presentation gives (3'), (4') and (5'), i.e., the formal relations which involve a_{n+1} have to be shown using the new relations with:

$$a_{n+1} = \sigma_n^{-1}..\sigma_3^{-1}a_3\sigma_3..\sigma_n.$$

We start by dealing with (3'): let $3 \leq i \leq n-1$. We compute:

$$\sigma_i a_{n+1} = \sigma_i \sigma_n^{-1}..\sigma_3^{-1} a_3 \sigma_3..\sigma_n = \sigma_n^{-1}..\sigma_3^{-1} \underbrace{\sigma_{i+1} a_3}_{(5)} \sigma_3..\sigma_n$$

$$= \sigma_n^{-1}..\sigma_3^{-1} a_3 \sigma_{i+1} \sigma_3..\sigma_n = \sigma_n^{-1}..\sigma_3^{-1} a_3 \sigma_3..\sigma_n \sigma_i$$

$$= a_{n+1} \sigma_i.$$

Moreover, $\sigma_2 a_{n+1} = \sigma_2 \sigma_n^{-1}..\sigma_3^{-1} a_3 \sigma_3..\sigma_n = \sigma_n^{-1}.. \underbrace{\sigma_2 \sigma_3^{-1} a_3 \sigma_3}_{(6'')} ..\sigma_n = \sigma_n^{-1}..\sigma_3^{-1} a_3 \sigma_3 \sigma_2..\sigma_n$

$$= a_{n+1} \sigma_2.$$

Hence, $\sigma_i a_{n+1} = a_{n+1} \sigma_i$ when $2 \leq i \leq n-1$. Thus, (3') is proved.

Since a_{n+1} and a_3 are conjugate, (3) gives directly (4').

Now we want to show that $\underbrace{\sigma_n a_{n+1} \sigma_n}_{:=x} = \underbrace{a_{n+1} \sigma_n a_{n+1}}_{:=y}$.

$$x = \sigma_{n-1}^{-1}..\sigma_3^{-1} a_3 \sigma_3..\sigma_n^2,$$

$$y = \sigma_n^{-1}..\sigma_3^{-1} a_3 \sigma_3..\sigma_{n-1} \sigma_n \sigma_{n-1}^{-1}..\sigma_3^{-1} a_3 \sigma_3..\sigma_n = \sigma_n^{-1}..\sigma_3^{-1} a_3 \sigma_n^{-1}..\sigma_4^{-1} \sigma_3 \sigma_4..\sigma_n a_3 \sigma_3..\sigma_n$$

$$= \sigma_n^{-1}..\sigma_3^{-1} \sigma_n^{-1}..\sigma_4^{-1} \underbrace{a_3 \sigma_3 a_3}_{(4)} \sigma_4..\sigma_n \sigma_3..\sigma_n = \sigma_n^{-1}..\sigma_3^{-1} \sigma_n^{-1}..\sigma_4^{-1} \sigma_3 a_3 \sigma_3 \sigma_4..\sigma_n \sigma_3..\sigma_n$$

$$= \sigma_{n-1}^{-1}..\sigma_3^{-1} \sigma_n^{-1}..\sigma_3^{-1} \sigma_3 a_3 \sigma_3 \sigma_4..\sigma_n \sigma_3..\sigma_n = \sigma_{n-1}^{-1}..\sigma_3^{-1} a_3 \sigma_n^{-1}..\sigma_4^{-1} \sigma_3 \sigma_4..\sigma_n \sigma_3..\sigma_n$$

$$= \sigma_{n-1}^{-1}..\sigma_3^{-1} a_3 \sigma_3..\sigma_{n-1} \sigma_n \sigma_{n-1}^{-1}..\sigma_3^{-1} \sigma_3..\sigma_n = \sigma_{n-1}^{-1}..\sigma_3^{-1} a_3 \sigma_3..\sigma_{n-1} \sigma_n^2$$

$$= x.$$

Finally, $F_n(a_3) = F_n(\sigma_3..\sigma_{n-1} a_n \sigma_{n-1}^{-1}..\sigma_3^{-1}) = \sigma_3..\sigma_{n-1} F_n(a_n) \sigma_{n-1}^{-1}..\sigma_3^{-1}$. This is equal to $\sigma_3..\sigma_n a_{n+1} \sigma_n^{-1}..\sigma_3^{-1}$, thus to a_3.

Now F_n would have the following form with the new presentation:

$$\begin{aligned} F_n : B(\widetilde{A_{n-1}}) &\longrightarrow B(\tilde{A}_n) \\ \sigma_i &\longmapsto \sigma_i \text{ for } 1 \leq i \leq n-1 \\ a_3 &\longmapsto a_3 \end{aligned}$$

Notice that we could have started with the group $B(\tilde{A}_1)$, which is a free group in two letters σ_1 and a_3, with a change in the sixth relation. On the other hand, it is obvious that $(B(\widetilde{A_{n+1}}), S')$ is not an Artin System.

2.4.2 $B(A_n)$ as a quotient of $B(\tilde{A}_n)$

Now we consider the element e in $B(\tilde{A}_n)$ given as

$$e = a_{n+1}^{-1}\sigma_n^{-1}\sigma_{n-1}^{-1}...\sigma_2^{-1}\sigma_1\sigma_2...\sigma_{n-1}\sigma_n = a_{n+1}^{-1}\sigma_1\sigma_2...\sigma_{n-1}\sigma_n\sigma_{n-1}^{-1}...\sigma_2^{-1}\sigma_1^{-1}.$$

Let N_e be the normal subgroup of $B(\tilde{A}_n)$ generated by e. Consider the quotient $Q = B(\tilde{A}_n)/N_e$.

Lemma 2.4.6. *In $B(\tilde{A}_n)$ (and in $B(A_n)$ as well), the element $b := \sigma_n^{-1}\sigma_{n-1}^{-1}...\sigma_2^{-1}\sigma_1\sigma_2...\sigma_{n-1}\sigma_n$ verifies the following relations:*

1) $\sigma_i b = b\sigma_i$ for $2 \leq i \leq n-1$,

2) $\sigma_n b\sigma_n = b\sigma_n b$,

3) $\sigma_1 b\sigma_1 = b\sigma_1 b$.

Proof. 1)

$$\sigma_i b = \sigma_i\sigma_n^{-1}\sigma_{n-1}^{-1}...\sigma_2^{-1}\sigma_1\sigma_2...\sigma_{n-1}\sigma_n = \sigma_n^{-1}...\underbrace{\sigma_i\sigma_{i+1}^{-1}\sigma_i^{-1}}_{\sigma_{i+1}^{-1}\sigma_i^{-1}\sigma_{i+1}}...\sigma_1\sigma_2...\sigma_{n-1}\sigma_n$$

$$= \sigma_n^{-1}\sigma_{n-1}^{-1}...\sigma_2^{-1}\sigma_1...\underbrace{\sigma_{i+1}\sigma_i\sigma_{i+1}}_{\sigma_i\sigma_{i+1}\sigma_i}..\sigma_1 = b\sigma_i.$$

3)

$$\sigma_n^{-1}\sigma_{n-1}^{-1}...\sigma_2^{-1}\sigma_1\sigma_2...\sigma_{n-1}\sigma_n\sigma_1\sigma_n^{-1}\sigma_{n-1}^{-1}...\sigma_2^{-1}\sigma_1\sigma_2...\sigma_{n-1}\sigma_n$$

$$= \sigma_n^{-1}\sigma_{n-1}^{-1}...\sigma_2^{-1}\underbrace{\sigma_1\sigma_2\sigma_1}_{\sigma_2\sigma_1\sigma_2}\sigma_2^{-1}\sigma_1\sigma_2...\sigma_{n-1}\sigma_n = \sigma_1^2\sigma_n^{-1}\sigma_{n-1}^{-1}...\sigma_3^{-1}\sigma_2\sigma_3...\sigma_{n-1}\sigma_n = \sigma_1 b\sigma_1.$$

In the same way we deal with (2), hence the proof is done.

□

We see directly that, when replacing b by a_{n+1} in this lemma, we get the defining relations of $B(\tilde{A_n})$ in which a_{n+1} is involved. Now the group Q is generated by the set $\{\sigma_1, \sigma_2..\sigma_n, a_{n+1}\}$ with the defining relations :

- $\sigma_i\sigma_j = \sigma_j\sigma_i$ where $1 \leq i, j \leq n$ when $|i-j| \geq 2$,
- $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ when $1 \leq i \leq n-1$,
- $\sigma_i a_{n+1} = a_{n+1}\sigma_i$ when $2 \leq i \leq n-1$,
- $\sigma_1 a_{n+1}\sigma_1 = a_{n+1}\sigma_1 a_{n+1}$ and $\sigma_n a_{n+1}\sigma_n = a_{n+1}\sigma_n a_{n+1}$. for $n \geq 2$,
- $a_{n+1} = \sigma_n^{-1}\sigma_{n-1}^{-1}...\sigma_2^{-1}\sigma_1\sigma_2...\sigma_{n-1}\sigma_n$.

Theorem 2.4.7. *The map $f : Q \longrightarrow B(A_n)$ defined by*

$$\sigma_i \longmapsto \sigma_i \text{ for } 1 \leq i \leq n,$$
$$a_{n+1} \longmapsto \sigma_n^{-1}\sigma_{n-1}^{-1}...\sigma_2^{-1}\sigma_1\sigma_2...\sigma_{n-1}\sigma_n,$$

is a group isomorphism.

Proof. By the lemma we see that f is a homomorphism, and it is surjective. Moreover, the following map:

$$g : B(A_n) \longrightarrow Q \text{ given by}$$
$$\sigma_i \longmapsto \sigma_i \text{ for } 1 \leq i \leq n,$$

is a group homomorphism, surjective indeed, for $\sigma_n^{-1}\sigma_{n-1}^{-1}...\sigma_2^{-1}\sigma_1\sigma_2...\sigma_{n-1}\sigma_n$ is send to itself, hence to a_{n+1}.

Obviously: $f \circ g = Id_{B(A_n)}$ and $g \circ f = Id_Q$, so f is indeed an isomorphism with $f^{-1} = g$.

□

With the notation as above:

$$B(\tilde{A_n}) \cong B(A_n) \ltimes N_e$$

Clearly this surjection respects the inclusion F_n, in other terms the following diagram commutes:

$$\begin{tikzcd}
B(\tilde{A_{n-1}}) \arrow[r, hook, "F_n"] \arrow[d, two heads, "\beta_{n-1}"'] & B(\tilde{A}_n) \arrow[d, two heads, "\beta_n"] \\
B(A_{n-1}) \arrow[r, hook, "x_n"] & B(A_n)
\end{tikzcd}$$

2.4.3 On the structure of $B(\tilde{A}_n)$: Schreier's theorem

Lemma 2.4.8. *Let G be a free group with $S = \{F_0, F_1, .., F_n\}$ as a set of free generators, let N be the subgroup:*

$$N = \left\{F_{i_1}^{\epsilon_1}..F_{i_j}^{\epsilon_j}..F_{i_m}^{\epsilon_m}; \sum \epsilon_j = 0\right\}$$

That is the normal group of words in which the number of positive letters is equal to the number of negative letters.

Then, N is an infinitely generated free group over the letters $F_0^j F_i F_0^{-(j+1)}$ where j is an arbitrary integer .

Proof. By Schreier's theorem we see that N is a free group. Now we apply Schreier's algorithm to show that $F_0^j F_i F_0^{-(j+1)}$ could be viewed as free generators, to do so we consider the following surjection:

$$\begin{aligned} G &\longrightarrow \langle F_0 \rangle \quad \text{given by:} \\ F_i &\mapsto F_0 \text{ for } 1 \leq i \leq n, \end{aligned}$$

where $\langle F_0 \rangle$ is to be the subgroup of G generated by F_0, that is the free group with one generator F_0. It is obvious that N is the kernel of this homomorphism (we could actually choose any of the letters of G to generate a range in order to have N as a kernel), hence

$$G \cong \langle F_0 \rangle \ltimes N.$$

So we can consider the set $\left\{F_0^j; j \in \mathbb{Z}\right\}$ as a full set of representatives of right cosets of N in G, thus it could be considered as a "Schreier's system". Now we define the following map:

$$\begin{aligned} \phi : G &\longrightarrow \langle F_0 \rangle \quad \text{given by:} \\ \phi(x) &= F_0^k \text{ when } x \in N F_0^k. \end{aligned}$$

By Schreier's theorem: $gs\phi(gs)^{-1}$ are free generators of N, where g runs over the "Schreier's system" and s runs over the set of free generators of G .

The element gs is of the form $F_0^j F_i$, while $\phi(gs)^{-1}$ is of the form:

$$\phi(F_0^j F_i)^{-1} = \phi(F_i^{-1} F_0^{-j}) = \phi(\underbrace{F_i^{-1} F_0^{-j} F_0^{j+1}}_{\in N} F_0^{-(j+1)}) = F_0^{-(j+1)}.$$

Thus $gs\phi(gs)^{-1}$ is of the form $F_0^j F_i F_0^{-(j+1)}$ where $0 \leq i \leq n$. The lemma follows.

$\square$

Now the following diagram commutes:

$$\begin{tikzcd}
B(\tilde{A}_n) \arrow[rr, two heads, "f"] \arrow[d, hook, "i_n"'] & & B(A_n) \\
B(B_{n+1}) \arrow[urr, two heads, "\alpha_n"'] & &
\end{tikzcd}$$

We can see that $ker(f) = ker(\alpha_n) \cap i_n(B(\tilde{A}_n))$. But $N_e = ker(f)$, so we are reduced to understanding the structure of $ker(\alpha_n)$. Set $G' := ker(\alpha_n)$.

Lemma 2.4.9. *[DG01]With the above notations, G' is a free subgroup of $B(B_{n+1})$ generated by the free generators F_i, where $0 \leq i \leq n$, $F_i = \sigma_i..\sigma_1 t \sigma_1^{-1}..\sigma_i^{-1}$ and $F_0 = t$.*

An element x is in N_e if and only if it is a word in G' and it is in $B(\tilde{A}_n)$, but since any element of $B(B_{n+1})$ is in $B(\tilde{A}_n)$ if and only if the sum of exponents of t is zero in an (every) expression of it, so we can apply the last lemma, taking G' for G. Then N_e is to be N, so our group N_e is an infinitely generated free group.

2.4.4 $B(A_{n+1})$, $B(B_n)$ and $B(\tilde{A}_n)$, arrows

In what follows we show the net of arrows between the three type of braid groups mentioned above. We are interested in investigating which arrows among those do respect the injections between groups of a given type in different number of generators. Roughly speaking: the arrows should be thought of as arrows defined over the "towers of groups", precisely the towers come from the injections between groups of the same types.

Consider x_n, the injection $B(A_{n-1}) \hookrightarrow B(A_n)$ mentioned in 1.2.1. Geometrically $B(B_n)$ embeds into $B(B_{n+1})$ by adding the $n+1$-th stand, that is:

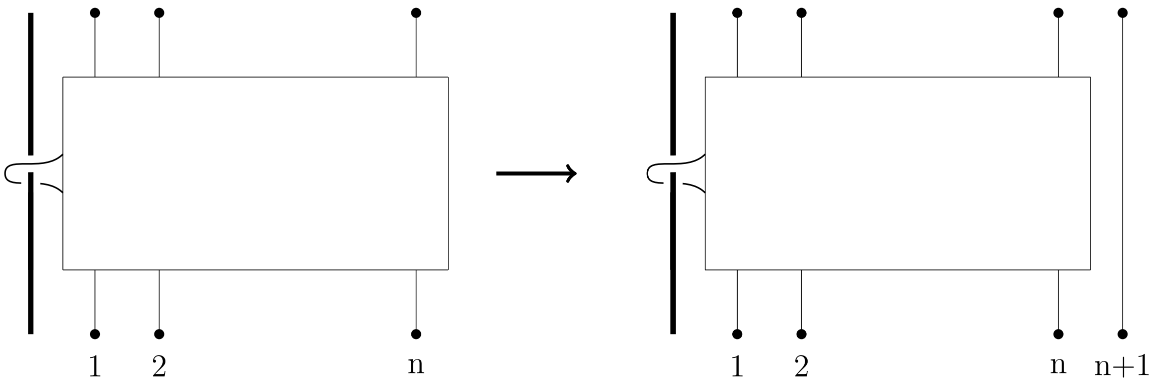


Figure 2.11: y_n

Let y_n be the injection $B(B_n) \hookrightarrow B(B_{n+1})$.

$B(A_n)$ injects in $B(B_{n+1})$ by sending σ_i to σ_i for $1 \leq i \leq n-1$, lets call this injection z_n. Take T to be the normal subgroup in $B(B_{n+1})$ generated by t, that is the subgroup generated by xtx^{-1} for $x \in B(B_{n+1})$. Obviously $B(B_{n+1})/T = B(A_n)$. in other words we have the following exact sequence $1 \to T \to B(B_{n+1}) \to B(A_n) \to 1$. Call α_n the surjection $B(B_{n+1}) \twoheadrightarrow B(A_n)$.

Geometrically $B(A_n)$ injects into $B(B_{n+1})$ by adding the first (fixed) strand, while $B(B_{n+1})$ surjects onto $B(A_n)$ by removing the very same strand

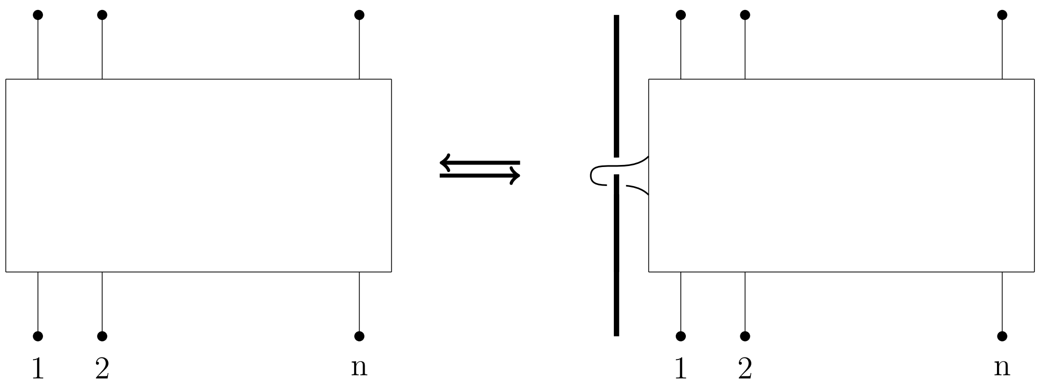


Figure 2.12: $B(A_n) \leftrightarrow B(B_{n+1})$

We get the following commutative diagrams:

$$\begin{tikzcd}
B(B_n) \arrow[r, hook, "y_n"] & B(B_{n+1}) \\
B(A_{n-1}) \arrow[u, hook, "z_{n-1}"] \arrow[r, hook, "x_n"'] & B(A_n) \arrow[u, hook, "z_n"']
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
B(B_n) \arrow[r, hook, "y_n"] \arrow[d, two heads, "\alpha_{n-1}"'] & B(B_{n+1}) \arrow[d, two heads, "\alpha_n"] \\
B(A_{n-1}) \arrow[r, hook, "x_n"'] & B(A_n)
\end{tikzcd}$$

(1) (2)

Diagram.1 commutes obviously, while for diagram.2 it is clear that

$$x_n\alpha_{n-1}(\sigma_i) = \alpha_n y_n(\sigma_i) \text{ for } 1 \leq i \leq n-1,$$
$$x_n\alpha_{n-1}(t) = 1 = \alpha_n y_n(t).$$

The embedding of $B(\tilde{A}_{n-1})$ into $B(\tilde{A}_n)$ is not as obvious as the other two embeddings. Consider the injection $y_n : B(B_n) \hookrightarrow B(B_{n+1})$ (see [GL97]), which sends a_n to $t\sigma_1..\sigma_{n-2}\sigma_{n-1}\sigma_{n-2}^{-1}..\sigma_1^{-1}t^{-1}$, which is equal to

$$\begin{aligned}
t\sigma_1..\sigma_{n-2}\sigma_{n-1}\underbrace{\sigma_n\sigma_{n-1}\sigma_{n-1}^{-1}\sigma_n^{-1}}_{1}\sigma_{n-2}^{-1}..\sigma_1^{-1}t^{-1} &= t\sigma_1..\sigma_{n-2}\underbrace{\sigma_{n-1}\sigma_n\sigma_{n-1}}\sigma_{n-1}^{-1}\sigma_n^{-1}\sigma_{n-2}^{-1}..\sigma_1^{-1}t^{-1} \\
&= t\sigma_1..\sigma_{n-2}\sigma_n\sigma_{n-1}\sigma_n\sigma_{n-1}^{-1}\sigma_n^{-1}\sigma_{n-2}^{-1}..\sigma_1^{-1}t^{-1} \\
&= \sigma_n\underbrace{t\sigma_1..\sigma_{n-1}\sigma_n\sigma_{n-1}^{-1}..\sigma_1^{-1}t^{-1}}_{a_{n+1}}\sigma_n^{-1}.
\end{aligned}$$

In other terms $y_n(a_n) = a_{n+1}$ which is in $B(\tilde{A}_n)$, in other terms the restriction of y_n to $B(\tilde{A}_{n-1})$ is equal to F_n as defined in 2.4.5, thus, this restriction is injective:

$$\begin{aligned}
F_n : B(\tilde{A}_{n-1}) &\longrightarrow B(\tilde{A}_n) \\
\sigma_i &\longmapsto \sigma_i \text{ for } 1 \leq i \leq n-1 \\
a_n &\longmapsto \sigma_n a_{n+1}\sigma_n^{-1}.
\end{aligned}$$

Set I_n to be the injection $B(A_n) \hookrightarrow B(\tilde{A}_n)$. Set i_n to be the injection $B(\tilde{A}_n) \hookrightarrow B(B_{n+1})$. We have the following commutative diagram:

$$
\begin{tikzcd}
B(B_n) \arrow[rrr, hook, "y_n"] & & & B(B_{n+1}) \\
& B(\tilde{A_{n-1}}) \arrow[ul, hook', "i_{n-1}"'] \arrow[r, hook, "F_n"] & B(\tilde{A}_n) \arrow[ur, hook, "i_n"] & \\
B(A_{n-1}) \arrow[uu, hook, "z_{n-1}"] \arrow[ur, hook, "I_{n-1}"] \arrow[rrr, hook, "x_n"] & & & B(A_{n-1}) \arrow[uu, hook', "z_n"'] \arrow[ul, hook', "I_n"']
\end{tikzcd}
$$

Geometrically, we realize in $B(\tilde{A}_n)$ the generators σ_i of $B(\tilde{A_{n-1}})$ (for $1 \leq i \leq n-1$) in the natural way, while concerning a_n we see that

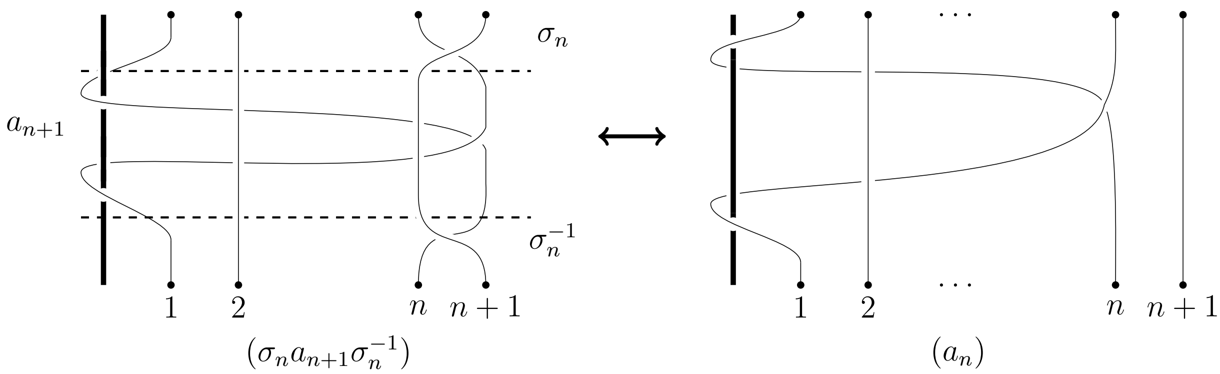


Figure 2.13: F_n

Recall that $B(B_n)$ is generated by $\{\sigma_1, \sigma_2..\sigma_n, a_n, \phi_n\}$. Now we consider ϕ_n as an automorphism of $B(\tilde{A_{n-1}})$. We call ϕ_n the Dynkin automorphism of order n, since it generates a subgroup of $Aut(B(\tilde{A_{n-1}}))$ of order n. It shifts the generators of the Dynkin diagram one step counter clockwise ($\sigma_1 \mapsto \sigma_2 \mapsto \sigma_3$ ... $\mapsto \sigma_{n-1} \mapsto a_n \mapsto \sigma_1$). In order to simplify we call it the Dynkin automorphism, referring to it by ϕ when there is no ambiguity.

On the other hand we see that in $B(\tilde{A}_n)$ for $2 \leq i \leq n-1$:

$$
\begin{aligned}
\sigma_n \sigma_{n-1}..\sigma_1 a_{n+1} \sigma_i &= \sigma_{i-1} \sigma_n \sigma_{n-1}..\sigma_1 a_{n+1}, \\
\sigma_n \sigma_{n-1}..\sigma_1 a_{n+1} \sigma_1 &= a_n \sigma_n \sigma_{n-1}..\sigma_1 a_{n+1}, \\
\sigma_n \sigma_{n-1}..\sigma_1 a_{n+1} a_n &= \sigma_{n-1} \sigma_n \sigma_{n-1}..\sigma_1 a_{n+1}.
\end{aligned}
$$

The last equality comes from the fact that $a_{n+1}a_n = \sigma_n^{-1}a_n\sigma_n a_n = \sigma_n^{-1}\sigma_n a_n \sigma_n$, which is equal to $a_n\sigma_n = \sigma_n a_{n+1}$. Hence $\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}$ acts on the elements of $B(\tilde{A}_{n-1})$ exactly the way as ϕ_n^{-1} does in $B(B_n)$.

Definition 2.4.10. *In $B(\tilde{A}_{n-1})$ we call $\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}$ the dominating element, and we denote it by D_n. When there is no ambiguity we call it D.*

Remark 2.4.11. *We show in 3.5 why we chose such a name for D_n and in which way it "dominates" the elements of $B(\tilde{A}_n)$ "with respect to" $B(\tilde{A}_{n-1})$.*

The following diagram commutes

$$\begin{tikzcd}
B(B_n) \arrow[drr, "f_n"] & & \\
B(\tilde{A}_{n-1}) \arrow[u, hook, "i_{n-1}"] \arrow[rr, hook, "F_n"'] & & B(\tilde{A}_n)
\end{tikzcd}$$

$$\begin{aligned}
\text{where } f_n : B(B_n) &\longrightarrow B(\tilde{A}_n) \\
\sigma_i &\longmapsto \sigma_i \text{ for } 1 \leq i \leq n-1 \\
a_n &\longmapsto \sigma_n a_{n+1}\sigma_n^{-1} \\
\phi_n &\longmapsto D_n^{-1}.
\end{aligned}$$

We consider the group $B(\tilde{A}_n)$ modulo the action of $\langle \phi_{n+1} \rangle_{Aut(B(\tilde{A}_n))}$ (the subgroup of $Aut(B(\tilde{A}_n))$ of order $n+1$ generated by the Dynkin automorphism). This group is isomorphic to the free group in one letter.

We have seen that $B(\tilde{A}_n)$ surjects onto $B(A_n)$. Call this surjection β_n, we get the following diagram:

$$\begin{tikzcd}
B(\tilde{A}_{n-1}) \arrow[rrr, hook, "F_n"] \arrow[dd, hook, "i_{n-1}"'] \arrow[dr, two heads, "\beta_{n-1}"] & & & B(\tilde{A}_n) \arrow[dd, hook, "i_n"] \arrow[dl, two heads, "\beta_n"'] \\
& B(A_{n-1}) \arrow[r, hook, "x_n"] & B(A_n) & \\
B(B_n) \arrow[rrr, hook, "y_n"] \arrow[ur, two heads, "\alpha_{n-1}"] & & & B(B_{n+1}) \arrow[ul, two heads, "\alpha_n"']
\end{tikzcd}$$

We have $\alpha_{n-1}i_{n-1}(a_{n-1}) = \alpha_{n-1}(t\sigma_1..\sigma_{n-2}\sigma_{n-1}\sigma_{n-2}^{-1}..\sigma_1^{-1}t^{-1}) = \sigma_1..\sigma_{n-2}\sigma_{n-1}\sigma_{n-2}^{-1}..\sigma_1^{-1}$, which is equal to $\beta_{n-1}(a_{n-1})$. Hence $\alpha_{n-1}i_{n-1} = \beta_{n-1}$. But we know already that $i_nF_n = y_ni_{n-1}$ and $x_n\alpha_{n-1} = \alpha_ny_n$. Hence $x_n\beta_{n-1} = \beta_nF_n$.

Finally we present the arrows shown earlier by the following two diagrams:

$$\begin{tikzcd}
& B(B_n) \arrow[rr, hook] & & B(B_{n+1}) \\
B(\tilde{A}_{n-1}) \arrow[ur, hook] \arrow[rr, hook] & & B(\tilde{A}_n) \arrow[ur, hook] & \\
& B(A_{n-1}) \arrow[rr, hook] \arrow[uu, hook] \arrow[ul, hook] & & B(A_n) \arrow[uu, hook] \arrow[ul, hook]
\end{tikzcd}$$

$$\begin{tikzcd}
& B(B_n) \arrow[rr, hook] \arrow[dd, two heads] & & B(B_{n+1}) \arrow[dd, two heads] \\
B(\tilde{A}_{n-1}) \arrow[ur, hook] \arrow[rr, hook] \arrow[dr, two heads] & & B(\tilde{A}_n) \arrow[ur, hook] \arrow[dr, two heads] & \\
& B(A_{n-1}) \arrow[rr, hook] & & B(A_n)
\end{tikzcd}$$

2.4.5 Affine links and closures

In what follows we give some definitions and basic results in the theory of links, the classical well known results concerning invariants of links are to be mentioned briefly. The aim is to define the concept of "affine" links, defining dual concepts and conventions to those in the classical theory. Most of theorems and results here are well explained in the literature, and we will not give details.

Let C_i be a circle in $\mathbb{R}^3$, where $1 \leq i \leq n$. Let $C^n := \cup_i C_i$ be the disjoint union of those n circles. We call C^n a rough link. Take the isotopy class of C^n, call it C, we call C a circle link or simply a link. We consider here only the piecewise linear links. If we orient the circles forming C, we say that it is an oriented link. Notice that orienting a circle is independent of orienting another one, since they do not intersect. Inverting the orientation of one of the circles gives a different oriented link.

Roughly speaking, the problem of finding an invariant for the set of links in $\mathbb{R}^3$ is to give names to links, in such way that any two links which have the same "shape" (i.e., we can arrive to one from the other by pulling and pushing the circles forming a link without cutting, adding or omitting any of the circles) have the same name.

We recall the results of Alexander and Markov concerning braids and links.

Consider a braid b, that is: a an element of $B(A_n)$ for some $1 \leq n$. As we have $n+1$ stand with $n+1$ points at the top (the same at the bottom), a path from a point (say the i-th point) at the top to the i-th at the bottom makes the i-th strand turns into a deformation of a circle in $\mathbb{R}^3$, repeating the same step with all the points (using non crossing paths) gives a union of disjoint deformed circles, hence a link. Thus we have defined a mapping from $\bigcup_{1 \leq i} B(A_i)$ into the set of links in $\mathbb{R}^3$. We call the image of b: the closure of b, denoted by $\hat{b}$.

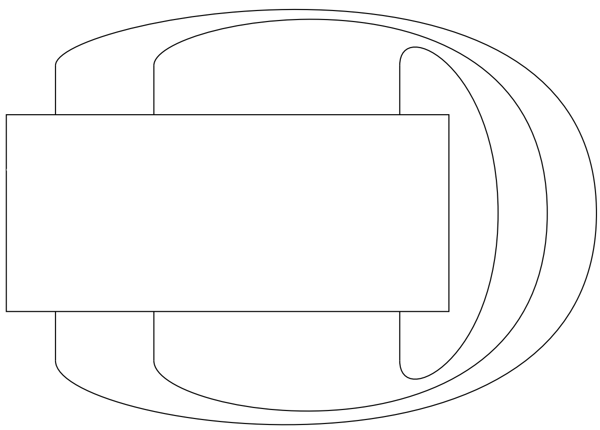


Figure 2.14:

We are interested in a map in the opposite direction.

Theorem 2.4.12. *(Alexander) Suppose that C is an oriented link, then there exists an integer $1 \leq n$ and b in $B(A_n)$ such that $\hat{b} = C$.*

In other terms, set OK to be the set of all oriented links in $\mathbb{R}^3$. Then the following map is surjective:

$$\begin{aligned} \bigcup_{1 \leq i} B(A_i) &\longrightarrow OK \\ b &\longmapsto \hat{b}. \end{aligned}$$

Let us present the main idea of the proof. Take any link C. Suppose it is the union of n deformed circles. We project it on $\mathbb{R}^2$ respecting the crossing points (the concept of positive and negative crossing makes it doable). We take any point in $\mathbb{R}^2$, say P. We take an orientation of every circle (by arbitrary orientation of those circles we get all the possible orientations of C). The point P defines negative and positive rounds, say that the negative, for example, can be shifted to the 'right' of P the positive on the 'left', hence we arrive to some presentation of C as the following

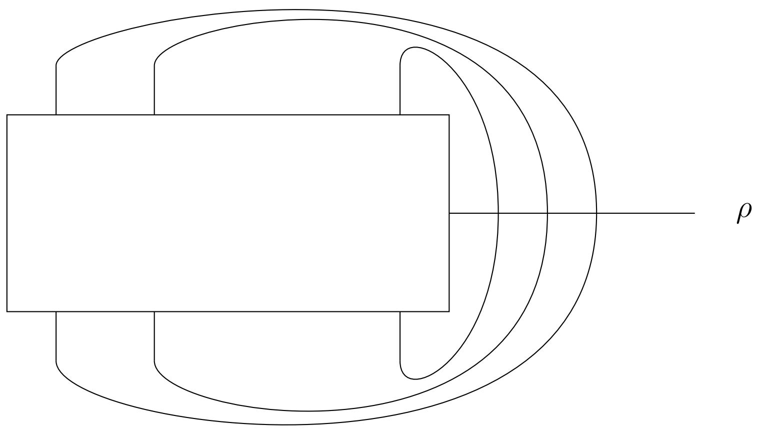


Figure 2.15:

Then we can cut the circle on the axis ρ, getting a braid whose closure is C.

The next question is, obviously: when do two braids have the same closure? The answer is a theorem of Markovs'.

Theorem 2.4.13. * *(Markov) Two braids have the same closure if and only if the exists an integer $1 \leq n$ such that: starting from one of the two braids we can arrive to the other by a finite number of transformations of the two following types:*

- $ab \leftrightarrow ba$ *Where a and b are in $B(A_n)$,*
- $a \leftrightarrow x\sigma_n$ *Where a is in $B(A_{n-1})$.*

After this theorem a very elegant answer would be to find a family of applications t_{n+1} defined over $B(A_n)$ such that for all $1 \leq n$:

- $t_n(ab) = t_n(ba)$ for all a and b are in $B(A_{n-1})$,
- $t_{n+1}(x\sigma_n) = t_n(x)$ for any x is in $B(A_{n-1})$.

The answer given by Jones was exactly of this form, it will be explained in details in the following sections. Here we attempt to define an "affine link" as a result of closing an affine braid, where we mean by affine braid an element of an $\tilde{A}$-type braid group. This task is not as evident as the A-type braid group, for we have many geometrical presentations of $\tilde{A}$-type braid groups. A choice must be made here, this is what we are about to do in the rest of this section.

In order to simplify we call an oriented link simply a link in $\mathbb{R}^3$ (in the literature S^3 is often used).

We call a B-braid any element in a given B-type braid. Clearly any affine braid is a B-braid, which have a presentation as a cylindrical braid which could be closed at least in two ways. Here we view B-braids as we did above, braids with one fixed strand. Now we consider the following application:

$$\begin{array}{rcl} {}_nI : B(B_n) & \longrightarrow & B(A_n) \\ \sigma_i & \mapsto & \sigma_{i+1} \text{ for } 1 \leq i \leq n-1 \\ t & \mapsto & \sigma_1^2. \end{array}$$

It is geometrically presented as follows:

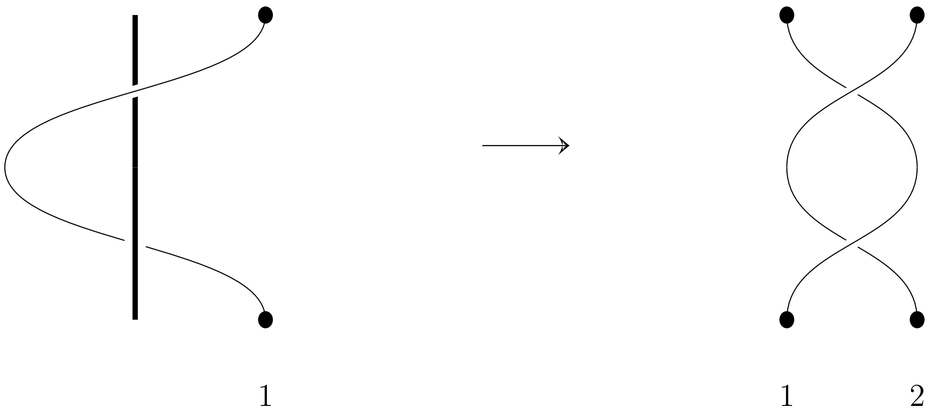


Figure 2.16: $t \longmapsto \sigma_1^2$

We see that

$$
\begin{aligned}
{}_nI(t\sigma_1 t\sigma_1) = \sigma_1^2\sigma_2\sigma_1^2\sigma_2 &= \underbrace{\sigma_1\sigma_2\sigma_1}\,\underbrace{\sigma_2\sigma_1\sigma_2} \\
&= \sigma_2\sigma_1\,\underbrace{\sigma_2\sigma_1\sigma_2}\,\sigma_1 = \sigma_2\sigma_1^2\sigma_2\sigma_1^2 =_n I(\sigma_1 t\sigma_1 t).
\end{aligned}
$$

In other terms ${}_nI$ is a homomorphism. Moreover, it is a monomorphism (see [Cri99]) . The following diagram, of injections, is commutative:

$$
\begin{tikzcd}
B(B_n) \arrow[rr, hook] & & B(A_n) \\
B(A_{n+1}) \arrow[u, hook, "\bar{z_n}", "\sigma_i \longmapsto \sigma_{i+1}"'] \arrow[urr, hook, "\bar{x_n}", "\sigma_i \longmapsto \sigma_{i+1}"'] & &
\end{tikzcd}
$$

Figure 2.17:

Here $\bar{z_{n-1}}$ and $\bar{x_n}$ are clearly injections, since z_{n-1} and x_n are so.

Corollary 2.4.14. * ***Any B-braid (hence affine) can be viewed as a braid in some A-type group. So, we can define the closure of an affine braid as the closure of its image under $_nI$. This injection means that any condition forcing any two affine braids to have the same closure is a consequence of the two Markov conditions.***

Proposition 2.4.15. *Let x be any affine braid (then) in $B(\tilde{A_{n-1}})$ for some $2 \leq n$. Then:*

1) *Given y in $B(\tilde{A_{n-1}})$ such that $\phi_n(y) = x$ then $\hat{y} = \hat{x}$ (in other terms $\hat{}$ is invariant under the action of the Dynkin automorphism).*

2) $x\hat{a_{n+1}} = \hat{x}$.

Proof. Suppose $\phi_n(y) = x$. That is equivalent to saying that $D_n x D_n^{-1} = y$ in $B(\tilde{A}_n)$. But by the first move $D_n x D_n^{-1} \leftrightarrow x D_n^{-1} D_n = x$. Thus $x \leftrightarrow y$.

On the other hand $xa_{n+1} = x\sigma_n^{-1}a_n\sigma_n = xa_n\sigma_n a_n^{-1} \leftrightarrow a_n^{-1}xa_n\sigma_n$, by the first move. But $a_n^{-1}xa_n\sigma_n \leftrightarrow a_n^{-1}xa_n$, by the second move. Hence we are reduced to $a_n^{-1}xa_n \leftrightarrow xa_na_n^{-1} = x$, by the first move, which means $xa_{n+1} \leftrightarrow x$.

□

Set $\widehat{B(A_n)}$ to be $\left\{\hat{b}; b \in B(A_n)\right\}$.

Now we reformulate our description what we called "B-links", defined as the closures of B-braids. In $\mathbb{R}^3$ the B-links are those links in which there is an oriented unknotted fixed circle. Now we can talk about $\widehat{B(A_n)}$ defined above without ambiguity (so as for $\widehat{B(\tilde{A}_n)}$). It is clear geometrically, that considering the fixed circle as a circle among the others gives the way in which $\widehat{B(B_n)}$ is contained in $\widehat{B(A_n)}$, while links in which there is no strings around the fixed circle, gives the inclusion of $\widehat{B(A_{n-1})}$ in $\widehat{B(A_n)}$. It is well known that B-links represent the links in a solid torus, the string which make the round around the hole of the torus represent $\hat{t}$ or $\hat{t^{-1}}$, which depends of course the orientation of the string.

In the same spirit we see that affine links are B-links in which the number of positive rounds equal the number of negative rounds, off course around the fixed circle. links who do not make rounds are counted here, actually they describe $\widehat{B(\tilde{A}_n)}$ containing $\widehat{B(A_n)}$.

Corollary 2.4.16. *Suppose that L_1 and L_2 are affine links. Suppose that l_1 and l_2 are two affine braids such that $\widehat{l_1} = L_1$ and $\widehat{l_2} = L_2$. Then L_1 and L_2 are isotopic if and only if l_1 and l_2 are equivalent by the sense of Markov, when being viewed in $\cup_{1\leq i} B(B_i)$, that is: if and only if $\bar{x_n}(l_1)$ and $\bar{x_n}(l_2)$ are equivalent.*

Proof. See theorem 5.2 in [GL97] with proposition 2.4.15 .

□

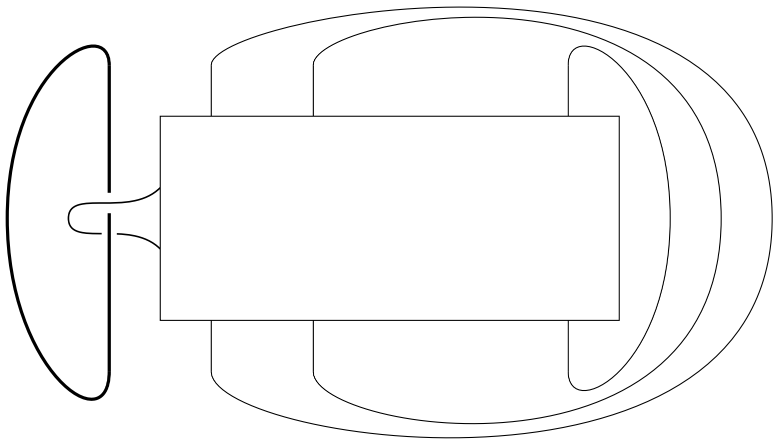


Figure 2.18: Affine closure

2.5 Bibliographical remarks and problems

Definitions and results of 2.1 and 2.2 are taken mostly from [Par07], while those of 2.3 are taken from [GL03], where we can see more details about the group $B(B_{n+1})$.

In lemma 2.4.9 we followed [DG01], in which section 5 and proposition 16 ensures us that the kernel of α_n is the group generated by Fj for $0 \leq j \leq n$, what is more that this group is free on the letters Fj.

Theorem 2.4.13 Announced by Markov himself, finally proven by Birman.

In corollary 2.4.14 we have resumed our conclusion about the closure of affine braids, one can view it as the "affine" version of theorem 2.4.13.

At the beginning we were all free,

then "they" invented: relations. Said a group.

فرادى ... كلنا أحرار.

Chapter

3 Coxeter groups

3.1 Coxeter systems *

Let S be a finite set, let $M = (m_{st})_{s,t\in S}$ be an associated Coxeter matrix, let $\Gamma = \Gamma(M)$ be its Dynkin graph. Consider the associated Artin System (B, S). As in definition 2.1.2, $B = B_\Gamma$ is to be the braid group of type Γ.

Definition 3.1.1. *We call the normal subgroup of $B = B_\Gamma$ generated by the set $\{s^2; s \in S\}$ the Γ-type pure braid group, it is to be denoted by $P = P_\Gamma$.*

Since P (generated as a group by the set $\{xs^{\pm 2}x^{-1};\ s \in S;\ x \in B\}$) is normal in B, we give the following definition.

Definition 3.1.2. *Put $W := W_\Gamma = B_\Gamma/P_\Gamma$. We call (W, S) a Coxeter System, and we call W a Γ-type Coxeter group.*

It should be known that up to isomorphism, there is a one-to-one bijection between the set of Coxeter Matrices and and the set of Coxeter Systems.

As a result from the definition we can see that if $\underline{S}$ is a formal copy of S, with $\underline{s}$ associated to s via this identification, then the natural surjection $B \twoheadrightarrow W$ sends s to $\underline{s}$. What is more, we have a presentation via generators and relations for the Γ-type Coxeter group.

The group $W = W_\Gamma$ is given by a set of generators $\underline{S} = \{\underline{s}; s \in S\}$ with as defining relations:

- $prod(\underline{s}, \underline{t} : m_{st}) = prod(\underline{t}, \underline{s} : m_{st})$ for any non-equal $\underline{s}, \underline{t}$ in $\underline{S}$ with $m_{s,t} \neq \infty$,
- $\underline{s}^2 = 1$ for any $\underline{s}$ in $\underline{S}$.

In what follows we identify $\underline{s}$ with s, when there is any confusion we are distinguishing one from the other by a suitable change of symbols, which is to be declared. Usually we refer to the first kind of relations by " braid relations", while the second is a style of "quadratic relations". In the literature, we often call the elements of $\underline{S}$ (say S from now on) simple reflections, and conjugations are called reflections. this comes from the fact that, as been mentioned in 2.1, the Coxeter group may be viewed as a real reflection group.

We say that a Coxeter group is irreducible if there does not exist two subsets in S (say S_1 and S_2) such that $\{S_1, S_2\}$ forms a partition of S, and every element in S_1 commutes with every element of S_2. Although most of the general results we are about to mention are independent of the irreducibility of our Coxeter group, we mean from now on by a Coxeter group an irreducible Coxeter group. Irreducible Coxeter groups are classified, but we are will not talk about the classification in this work. We give the presentations of the different types used in our work in this section, nevertheless, it was shown that a given Coxeter group is irreducible if and only if its Dynkin graph is connected (it is clear by definition that if we have a Dynkin graph Γ which is a disjoint union of two graphs, say Γ_1 and Γ_2, then any vertex of Γ_1 commutes with all the vertices of Γ_2).

The length of a word has the familiar meaning of the length in a group given by generators and relations, i.e., we consider the length with respect to S. One can show easily - using the fact that each one of the defining relations of a Coxeter group contains an even number of factors- that for every w in W and s in S we have $l(sw) = l(w) \pm 1$.

The next theorem (named after Matsumoto) which has many versions in literature, gives another definition of a Coxeter system, this explains the French sentence: Matsumoto et Coxeter, c'est la meme chose, presque!.

Theorem 3.1.3. *(Tits) Suppose that W is a group generated by a set S. Then the following two statements are equivalent:*

- *The pair (W, S) is a Coxeter System.*
- *For all $s \in S$, we have $s^2 = 1$, and if $s_1s_2..s_r$ is any reduced expression in elements of S, and if we have some $t \in S$, such that $ts_1s_2..s_r$ is not reduced, then there exists $1 \leq i \leq r$, such that $s_1s_2..s_r = ts_1..s_{i-1}s_{i+1}..s_r$.*

Theorem 3.1.4. *Suppose that (W, S) is a Coxeter system. Then any reduced expression for an element of W can be deduced from any other reduced expression, only by applying braid relations.*

This last theorem has a main role in describing the nature of elements of the structures related to a Coxeter group, for example we have the next result concerning braid groups.

Corollary 3.1.5. *Let Γ be a Dynkin graph. Let W be the Γ-type Coxeter group. Let B be the Γ-type braid group. Let w in W be given by some reduced expression $w = s_1 s_2..s_r$. Then the element $\overline{w}$ in B given by $\overline{w} = s_1 s_2..s_r$ is well defined,.i.e., it does not depend the reduced expression of w in W.*

Definition 3.1.6. *Let w be in W. We call the subset of S consisting of all generators appearing in a (any) reduced expression of w the support of w. It is to be denoted by $Supp(w)$.*

We define $\mathscr{L}(w)$ to be the set of $s \in S$ such that $l(sw) < l(w)$, in other terms s appears at the left edge of some reduced expression of w. Similarly we define $\mathscr{R}(w)$.

There is a class of subgroups which is of a great importance in the theory of Coxeter groups, we start by the following lemma.

Lemma 3.1.7. *Let (W, S) be a Coxeter system. Suppose that $I \subset S$. Set W_I to be the group generated by I. Then (W_I, I) is a Coxeter system, moreover, its Coxeter Matrix is the sub-matrix of that of W indexed by I.*

Definition 3.1.8. *With the notations above, W_I is to be defined as a parabolic subgroup of W.*

A Coxeter group is said to be of spherical type if it is of finite order, if not it is said to be of affine type.

3.2 $W(A_n)$ as a parabolic subgroup of $W(B_n)$

In this section we review briefly some definitions and basic facts about the two spherical A-type and B type Coxeter groups.

The A-type Coxeter group with n generators $W(A_n)$ is the quotient $B(A_n)/P(A_n)$. Where $P(A_n)$ is the pure braid group in $B(A_n)$. It is given by $S = \{\sigma_1, \sigma_2, .., \sigma_n\}$ as a set of generators with following defining relations:

- $\sigma_i\sigma_j = \sigma_j\sigma_i$ where $1 \leq i, j \leq n$ and $|i-j| \geq 2$,
- $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ where $1 \leq i \leq n-1$,
- $\sigma_i^2 = 1$ for all $1 \leq i \leq n$.

Remark 3.2.1. *As we did in the general definition of Coxeter groups, we will - with the three types of Coxeter groups treated in this work- keep the same notation for the generators of the braid groups and their images under the natural surjections, i.e., in this chapter σ_i (resp. a_i) are the images of our σ_i (resp. a_i) in the first chapter.*

Consider the group of permutations of a set with cardinal $n+1$ (say $\{1, 2.., n+1\}$). This group is called Sym_n. Call $f_i := (i, i+1)$ the permutation exchanging i and $i+1$ where $1 \leq i \leq n$, and fixing the other numbers. Now, Sym_{n-1} embeds in Sym_n by viewing any permutation of $\{1, 2.., n\}$ as a permutation of $\{1, 2.., n+1\}$ which fixes $n+1$.

Lemma 3.2.2. *[Bou81] With the above conventions, $\sigma_i \mapsto f_i$ for $1 \leq i \leq n$ define an isomorphism between Sym_n and $W(A_n)$ which respects the inclusion $Sym_{n-1} \hookrightarrow Sym_n$.*

Thus, it is obvious that the function which sends σ_i to σ_i for all $1 \leq i \leq n-1$ is an injection of $W(A_{n-1})$ into $W(A_n)$, call it x_{n-1}. We are however, interested in viewing $W(A_n)$ from a " Coxeter" point of view. The group $W(A_n)$ is of order $(n+1)!$, in which any element is either in $W(A_{n-1})$ or can be written as $u\sigma_n\sigma_{n-1}..\sigma_i$, where $u \in W(A_{n-1})$ and $1 \leq i \leq n$ (see [Bou81]). In other terms the set

$$\{1, \sigma_n\sigma_{n-1}..\sigma_i; 1 \leq i \leq n\}$$

is the set of distinguished right coset representatives of $W(A_{n-1})$ in $W(A_n)$. We deduce that the set $\{1, \sigma_n\}$ is the set of distinguished double coset representatives of $W(A_{n-1})$ in $W(A_n)$.

The same holds for B-type Coxeter group with $n+1$ generators $W(B_{n+1})$. It is the quotient $B(B_{n+1})/P(B_{n+1})$, where $P(B_{n+1})$ is the pure braid group in $B(B_{n+1})$ In other terms, it is the normal subgroup of $B(B_{n+1})$ generated by $\{t^2, \sigma_1^2, \sigma_2^2, .., \sigma_n^2\}$. It is given by $S = \{t, \sigma_1, \sigma_2, .., \sigma_n\}$ as a set of generators with defining relations:

- $\sigma_i\sigma_j = \sigma_j\sigma_i$ where $1 \leq i, j \leq n$, and $|i-j| \geq 2$,
- $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ where $1 \leq i \leq n-1$,
- $\sigma_i t = t\sigma_i$ when $2 \leq i \leq n$,
- $\sigma_1 t\sigma_1 t = t\sigma_1 t\sigma_1$,
- $t^2 = \sigma_i^2 = 1$ for all $1 \leq i \leq n$.

Set y_{n-1} to be the injection of $W(B_n)$ into $W(B_{n+1})$ (see [GL97]).

We set $t_i := \sigma_i\sigma_i..\sigma_i t\sigma_i..\sigma_i\sigma_i$ for $0 \leq i \leq n$, where $t_0 = t$. The the set of distinguished right coset representatives of $W(B_n)$ in $W(B_{n+1})$ is

$$\{1, t_n, \sigma_n\sigma_{n-1} \ldots \sigma_{n+1-k}, \sigma_n\sigma_{n-1} \ldots \sigma_{n+1-k}t_{n+1-k-1}; 1 \leq k \leq n\}.$$

We see directly that $\{1, \sigma_n, t_n\}$ is the set of distinguished double coset representatives of $W(B_n)$ in $B(A_{n+1})$ (see [GL97]). Notice that $W(B_n)$ (resp.$W(A_{n-1})$) is a parabolic subgroup of $W(B_{n+1})$ (resp. $W(A_n)$).

As we did in the first section with braid groups, we set T to be the normal subgroup in $W(B_{n+1})$ generated by t, which is the subgroup generated by xtx^{-1} for $x \in W(B_{n+1})$. Obviously $W(B_{n+1})/T = W(A_n)$. We get the following exact sequence $1 \to T \to W(B_{n+1}) \to W(A_n) \to 1$. We denote to the surjection $B(B_{n+1}) \twoheadrightarrow B(A_n)$ by α_n. We get the following commutative diagram

$$\begin{tikzcd}
W(B_n) \arrow[r, hook, "y_n"] \arrow[d, two heads, "\alpha_{n-1}"'] & W(B_{n+1}) \arrow[d, two heads, "\alpha_n"] \\
W(A_{n-1}) \arrow[r, hook, "F_n"'] & W(A_n)
\end{tikzcd}$$

Lemma 3.2.3. *[Bou81] Let $(\mathbb{Z}/2\mathbb{Z})$ be the group of order 2, then*

$$W(B_{n+1}) \cong (\mathbb{Z}/2\mathbb{Z})^{n+1} \rtimes W(A_n).$$

Set i_n to be the injection of $W(A_n)$ into $W(B_{n+1})$. We get the following commutative diagram (where every subgroup is parabolic):

$$\begin{tikzcd}
W(B_n) \arrow[r, hook, "y_n"] & W(B_{n+1}) \\
W(A_{n-1}) \arrow[r, hook, "F_n"'] \arrow[u, hook, "i_{n-1}"] & W(A_n) \arrow[u, hook, "i"']
\end{tikzcd}$$

3.3 The affine Coxeter group $W(\tilde{A}_n)$

In this section we present affine Coxeter groups in different ways. We use one of them to determine left and double classes and their representatives. Those representatives are not given by a reduced expression.

3.3.1 Presentations

As we defined the last two types of Coxeter groups, we define the $\tilde{A}$-type braid group with $n+1$ generators to be the quotient $B(\tilde{A}_n)/P(\tilde{A}_n)$, where $P(\tilde{A}_n)$ is the pure braid group in $W(\tilde{A}_n)$. Also $W(\tilde{A}_n)$ is the group presented by a set of generators $\{\sigma_1, \sigma_2..\sigma_n, a_{n+1}\}$, together with the following defining relations:

- $\sigma_i\sigma_j = \sigma_j\sigma_i$ where $1 \leq i, j \leq n$, and $|i-j| \geq 2$,
- $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ when $1 \leq i \leq n-1$,
- $\sigma_i a_{n+1} = a_{n+1}\sigma_i$ when $2 \leq i \leq n-1$,
- $\sigma_1 a_{n+1}\sigma_1 = a_{n+1}\sigma_1 a_{n+1}$,
- $\sigma_n a_{n+1}\sigma_n = a_{n+1}\sigma_n a_{n+1}$,
- $a_{n+1}^2 = \sigma_n^2 = 1$ for $1 \leq i \leq n$.

We give another presentation of $W(\tilde{A}_n)$. Let u be a permutation of $\mathbb{Z}$.

Definition 3.3.1. *u is said to be an m-periodic permutation if $u(i+m) = u(i)+m$ for any $i \in \mathbb{Z}$. We define the total shift of u (where u is any m-periodic permutation) to be:*

$$\frac{1}{m}\sum_{i=1}^{i=m}\Big(u(i)-i\Big).$$

We set ${}_0^m\mathbb{Z}$ to be the set of m-periodic permutations with total shift equal to 0. It forms a subgroup of the group of permutations of $\mathbb{Z}$. Let i be an integer such that $1 \leq i \leq m-1$. Then the m-periodic permutation which sends i to $i+1$ (and vice versa) and fixes the set $\{1, 2.., m\} - \{i, i+1\}$ is to be denoted by $s_i = (i, i+1)$. Now set $s_m := (1, m)$. Here ${}_0^m\mathbb{Z}$ is generated by $\{s_i; 0 \leq i \leq m\}$. It is clear that ${}_0^m\mathbb{Z}$ injects into ${}_0^{m+1}\mathbb{Z}$ by viewing any permutation v in ${}_0^m\mathbb{Z}$ as a permutation of ${}_0^{m+1}\mathbb{Z}$ which fixes $m+1$, it is clear that viewing v in such a way shifts its period from m to $m+1$, while the total shift of v is equal to

$\frac{1}{m+1}\sum\limits_{i=1}^{i=m+1}\left(v(i)-i\right)$, which is equal to

$$\frac{1+m}{m}\left[\frac{1}{m}\sum_{i=1}^{i=m}(u(i)-i)\right]+\frac{1}{m+1}(m+1-(m+1)),$$

which is obviously 0. We denote the generators of ${}_0^{m+1}\mathbb{Z}$ by t_i, where $1 \leq i \leq m+1$. We see that this injection sends s_i to t_i, for $1 \leq i \leq m-1$. Now we realize s_m as an expression of t_i. Considering the segment label from 1 to $m+1$ we see that in ${}_0^{m+1}\mathbb{Z}$ the permutation which replace 1 by m is equal to $(m, m+1) \circ (1, m+1) \circ (m, m+1)$. Hence by the injection above s_m is send to $t_m t_{m+1} t_m$.

Theorem 3.3.2. *[Lus83] With the above notation, we have ${}_0^{n+1}\mathbb{Z} \cong W(\tilde{A}_n)$. This isomorphism sends $(i, i+1)$ to σ_i, for $1 \leq i \leq n-1$, and sends $(1, n+1)$ to a_{n+1}.*

Now we see that $W(\tilde{A}_{n-1})$ is indeed injected into $W(\tilde{A}_n)$. We call this injection F_n. We have

$$\begin{aligned} F_n : W(\tilde{A}_{n-1}) &\longrightarrow W(\tilde{A}_n) \\ \sigma_i &\longmapsto \sigma_i \text{ for } 1 \leq i \leq n-1 \\ a_n &\longmapsto \sigma_n a_{n+1} \sigma_n. \end{aligned}$$

Remark 3.3.3. *Recall The presentation of $B(\tilde{A}_n)$ given in 2.4. It gives obviously, a new presentation of $W(\tilde{A}_n)$, since a_3 and a_{n+1} are conjugate, thus $a_3^2 = 1$ gives $a_{n+1}^2 = 1$, and vice versa . In other terms the relations (1), (2) ... (6) added to $a_3^2 = \sigma_i^2 = 1$ with the set $S' = \{\sigma_1, \sigma_2..\sigma_n, a_3\}$ gives a (generators and relations)-presentation of $W(\tilde{A}_n)$. the morphism F_n above, as in the braid group case, becomes*

$$\begin{aligned} F_n : B(\tilde{A}_{n-1}) &\longrightarrow B(\tilde{A}_n) \\ \sigma_i &\longmapsto \sigma_i \textit{ for } 1 \leq i \leq n-1 \\ a_3 &\longmapsto a_3. \end{aligned}$$

That is why we refer to it as the "parabolic-like" presentation, for the set of generators of $W(\tilde{A}_{n-1})$ is a subset of that of $W(\tilde{A}_n)$. The inclusion is given by restricting F_n to the set of generators of $W(\tilde{A}_{n-1})$. Unfortunately it is not Parabolic, for $(W(\tilde{A}_n), S')$ is not a Coxeter system. Yet, the relation $\sigma_3\sigma_2 a_3 \sigma_3 = \sigma_2 a_3 \sigma_3 \sigma_2$ is not a very "strange" relation! In other words, if one would like to extend the family of braid relations, to be relations involving three elements, this relation is a very good candidate. Moreover, the element a_3 is a conjugate to a simple reflection, hence it is a reflection. We see in some works of Lee, Birman and Bessis that sometimes viewing a Coxeter group as a group generated by *its reflection is a powerful tool. Here we are talking about viewing this group generated by a set of mixed elements : reflections and simple reflections! Finally we will see later that this presentation is valid for the other structures which will be treated in what follows.*

Now we explain a third way of presenting $W(\tilde{A}_n)$: as a semi-direct product. We start with the following lemme.

Lemma 3.3.4. *[GL03] Let $\mathbb{Z}_n$ be the free Abelian group with n generators. Then,*

$$W(\tilde{A}_n) \cong \mathbb{Z}_n \rtimes W(A_n).$$

In what follows we explain in details this semidirect product. Here $\mathbb{Z}_n$ can be viewed as $\oplus_{j=i}^{j=1} \langle \alpha_i \rangle$, where $\alpha_1, \alpha_2, .., \alpha_n$ are the following free generators:

$$\begin{aligned} \alpha_1 &= \sigma_1 a_2, \\ \alpha_2 &= \sigma_2 \alpha_1 \sigma_2 \ \ \dots\ , \\ \alpha_j &= \sigma_j \alpha_{j-1} \sigma_j, \text{ where } j = 2,\ \dots\ , n. \end{aligned}$$

The group $\mathbb{Z}_n$ can be viewed as $\left\{(z_0, z_1,\ \dots\ , z_n) \in \mathbb{Z}^{n+1}; \sum_{i=0}^{i=n} z_i = 0\right\}$. The action of $\sigma_j \in A_n$ on $(z_0, z_1,\ \dots\ , z_n)$ permutes z_{j-1} and z_j. Now we describe the law of the semi-direct product of $W(A_n)$ by $\mathbb{Z}_n$ (being written additively). We have:

$$\Big((z_0, z_1,\ \dots\ , z_n), \sigma_i\Big).\Big((\zeta_0, \zeta_1,\ \dots\ , \zeta_n), \sigma_j\Big) = \Big((z_0, z_1,\ \dots\ , z_n) + \sigma_i\big[(\zeta_0, \zeta_1,\ \dots\ , \zeta_n)\big], \sigma_i\sigma_j\Big).$$

The generators are presented as follows:

$$\begin{aligned} \alpha_1 &= (-1, 1, 0, 0,\ \dots\ , 0), \\ \alpha_2 &= (-1, 0, 1, 0,\ \dots\ , 0), \\ &. \\ &. \\ &. \\ \alpha_n &= (-1, 0,\ \dots\ ., 0, 1). \end{aligned}$$

Notice that $\alpha_j = \sigma_j \alpha_{j-1} \sigma_j$, except for σ_1 which sends α_1 to $-\alpha_1$ (where $\mathbb{Z}_n$ is written additively). Any element x of $W(\tilde{A}_n)$ can be written uniquely as $x_n z_n$ where $x_n \in W(A_n)$ and $z_n \in \mathbb{Z}_n$. We denote the image of z_n under x_n by $[z_n]_{x_n}$. Thus $x_n z_n = [z_n]_{x_n} x_n$. The same holds for an element y in $W(\tilde{A}_n)$: it can be written in a unique way in the form $\eta_n y_n$ where $y_n \in W(A_n)$ and $\eta_n \in \mathbb{Z}_n$. On the semi-direct product level, we see that

$$x_n z_n = (0, x_n).(z_n, 1) = \Big([z_n]_{x_n}, x_n\Big) = [z_n]_{x_n} x_n,$$

$$\text{while } \eta_n y_n = (\eta_n, 1).(0, y_n) = (\eta_n, y_n).$$

$$\text{Moreover, we have } \eta_n y_n = y_n y_n^{-1} \eta_n y_n = y_n [\eta_n]_{y_n^{-1}} y_n^{-1} y_n = y_n [\eta_n]_{y_n^{-1}}.$$

Now we consider some examples. We start by a_2, which is written as $a_2 = \sigma_1\alpha_1$, which is equal to $(0,\sigma_1)(\alpha_1,1) = ([\alpha_1]_{\sigma_1},\sigma_1) = (-\alpha_1,\sigma_1)$. On the other hand we have:

$$\begin{aligned}
a_2 &= \sigma_2\sigma_3 \,...\, \sigma_n a_{n+1}\sigma_n \,...\, \sigma_3\sigma_2,\\
\text{thus, } a_{n+1} &= \sigma_n\sigma_{n-1} \,...\, \sigma_2 a_2 \sigma_2 \,...\, \sigma_{n-1}\sigma_n\\
&= (0,\sigma_n\sigma_{n-1} \,...\, \sigma_2)(-\alpha_1,\sigma_1)(0,\sigma_2 \,...\, \sigma_{n-1}\sigma_n)\\
&= \Big([-\alpha_1]_{\sigma_n\sigma_{n-1} \,...\, \sigma_2},\sigma_n\sigma_{n-1} \,...\, \sigma_2\sigma_1\Big)\Big(0,\sigma_2 \,...\, \sigma_{n-1}\sigma_n\Big)\\
&= (-\alpha_n,\sigma_n\sigma_{n-1} \,...\, \sigma_2\sigma_1)(0,\sigma_2 \,...\, \sigma_{n-1}\sigma_n).\\
\text{Hence } a_{n+1} &= (-\alpha_n,\sigma_n \,...\, \sigma_2\sigma_1\sigma_2 \,...\, \sigma_n),\\
\text{in other terms: } a_{n+1} &= \alpha_n^{-1}\sigma_n \,...\, \sigma_2\sigma_1\sigma_2 \,...\, \sigma_n.
\end{aligned}$$

For a_n, which equals $\sigma_n a_{n+1}\sigma_n = (0,\sigma_n)(-\alpha_n,\sigma_n \,...\, \sigma_2\sigma_1\sigma_2 \,...\, \sigma_n)(0,\sigma_n)$, we have:

$$a_n = \Big([-\alpha_n]_{\sigma_n},\sigma_{n-1} \,...\, \sigma_2\sigma_1\sigma_2 \,...\, \sigma_{n-1}\sigma_n\Big)\Big(0,\sigma_n\Big) = \Big(-\alpha_{n-1},\sigma_{n-1} \,...\, \sigma_2\sigma_1\sigma_2 \,...\, \sigma_{n-1}\Big).$$

Now we reconsider the surjection in 3.3.1,

$$\pi_n : B(\tilde{A}_n) \longrightarrow W(\tilde{A}_n).$$

We reconsider the subgroup N_e of $B(\tilde{A}_n)$ as well (which is the normal subgroup generated by e, where $e = a_{n+1}\sigma_n^{-1}\sigma_{n-1}^{-1} \,...\, \sigma_2^{-1}\sigma_1\sigma_2 \,...\, \sigma_{n-1}\sigma_n)$. Now N_e is generated by the elements geg^{-1} where g is in $B(\tilde{A}_n)$. We take any x in N_e. Then, there is a non negative r such that $x = g_1e^{\pm1}g_1^{-1} \,...\, g_re^{\pm1}g_r^{-1}$. By noticing that $\pi_n(e) = \alpha_n^{-1}$, where α_n is the very α_n in $\mathbb{Z}_n$ above, we see that

$$\pi_n(x) = \underbrace{\pi_n(g_1)\alpha_n(\pi_n(g_1))^{-1}}_{\in\mathbb{Z}_n} \;...\; \underbrace{\pi_n(g_r)\alpha_n(\pi_n(g_r))^{-1}}_{\in\mathbb{Z}_n}, \text{ since } \mathbb{Z}_n \text{ is normal in } W(\tilde{A}_n).$$

In other terms:

$$\pi_n(N_e) \subseteq \mathbb{Z}_n.$$

On the other hand, we can lift the generators of $\mathbb{Z}_n$ to conjugates of e. Namely we set in $B(\tilde{A}_n)$:

$$\begin{aligned}
\widehat{\alpha}_n &:= e^{-1},\\
\widehat{\alpha}_{n-1} &:= \sigma_n\widehat{\alpha}_n\sigma_n^{-1},\\
&\;\;\vdots\\
\text{and } \widehat{\alpha}_1 &:= \sigma_2\widehat{\alpha}_2\sigma_2^{-1}.
\end{aligned}$$

So, for $1 \leq i \leq n-1$, we have:

$$\widehat{\alpha}_i = \sigma_i..\sigma_n\widehat{\alpha}_n\sigma_n^{-1}..\sigma_i^{-1}.$$

We can directly see that $\pi_n(\widehat{\alpha}_i) = \alpha_i$, for $1 \leq i \leq n$. in other terms: the restriction of π_n to N_e (say φ) surjects onto $\mathbb{Z}_n$. One can see that $ker(\varphi) = ker(\pi_n) \cap \tilde{P}_n$.

Thus, the surjection π_n respects the two semi-direct products in the following sense: when passing from the affine braid group to affine Coxeter group, by "quotienting" on the affine pure braid group, the braid group $B(A_n)$ turns into the Coxeter group $W(A_n)$, while the free group N_e turns into the free Abelian group $\mathbb{Z}_n$:

$$\begin{tikzcd}
B(\tilde{A}_n) \arrow[d, two heads, "\pi_n"] & \simeq & Ne \arrow[d, two heads, "\pi_n|_{Ne}"] & \rtimes & B(A_n) \arrow[d, two heads, "\pi_n|_{B(A_n)}"] \\
W(\tilde{A}_n) & \simeq & Z_n & \rtimes & W(A_n)
\end{tikzcd}$$

3.3.2 Left and double classes of $W(\tilde{A_{n-1}})$ in $W(\tilde{A}_n)$

Proposition 3.3.5. *Let L_n be the set of distinguished left coset representatives of $W(\tilde{A_{n-1}})$ in $W(\tilde{A}_n)$. In $W(\tilde{A}_n)$ we set $\sigma_{n+1} = 1$. Then, we have*

$$L_n = \left\{\sigma_r..\sigma_n\alpha_n^k;\ (k \in \mathbb{Z}), (1 \leq r \leq n+1)\right\}.$$

Proof. Let x and y be in $W(\tilde{A}_n)$. Then x is written uniquely as $x = x_n\alpha_n^k z_{n-1}$, and the same holds for y, say it is written as $y = y_n\alpha_n^h z'_{n-1}$. Here x_n and y_n are in $W(A_n)$, while z_{n-1} and z'_{n-1} are in $\mathbb{Z}_{n-1}$ and k, h are two integers.

$$\text{Now, } xW(\tilde{A_{n-1}}) = x_n\alpha_n^k W(A_{n-1})\mathbb{Z}_{n-1},$$
$$\text{and } yW(\tilde{A_{n-1}}) = y_n\alpha_n^h W(A_{n-1})\mathbb{Z}_{n-1}.$$

Suppose that $xW(\tilde{A_{n-1}}) = yW(\tilde{A_{n-1}})$. Then, we have:

$x_n\alpha_n^k A_{n-1}\mathbb{Z}_{n-1} = y_n\alpha_n^h A_{n-1}\mathbb{Z}_{n-1} \Rightarrow$ there exists $w \in W(A_{n-1})$ and $\eta_{n-1} \in \mathbb{Z}_{n-1}$, such that $x_n\alpha_n^k = y_n\alpha_n^h\eta_{n-1}w$. But $y_n\alpha_n^h\eta_{n-1}w = y_n w w^{-1}\alpha^h\eta_{n-1}w$.

Since $\mathbb{Z}^n$ is normal in $W(\tilde{A}_n)$, it is straightforward that $w^{-1}\alpha^h\eta_{n-1}w$ is in $\mathbb{Z}^n$. In other terms $x_n\alpha_n^k = y_n w \underbrace{w^{-1}\alpha^h\eta_{n-1}w}$.

Since the writing is unique, we get the two following equalities:

(1) $x_n = y_n w$,

(2) $\alpha_n^k = w^{-1}\alpha^h \eta_{n-1} w$.

(1) Means that $x_n W(A_{n-1}) = y_n W(A_{n-1})$, in other words x_n, y_n belong to the same left class of $W(A_{n-1})$ in $W(A_n)$. This class has as a representative one of the elements of the set $\{\sigma_i..\sigma_{n-1}\sigma_n;\ i = 1, 2, ..n+1\}$.

(2) Gives that $\alpha_n^h w \alpha_n^{-k} w^{-1} \in \mathbb{Z}_{n-1}$. That is:

$$(h\alpha_n, 1)(0, w).(-k\alpha_n, 1).(0, w^{-1}) \in \mathbb{Z}_{n-1}.$$

This gives:

$$(h\alpha_n, w)(-k\alpha_n, w^{-1}) \in \mathbb{Z}_{n-1}.$$

Hence, $(h\alpha_n - k[\alpha_n]_w, 1) \in \mathbb{Z}_{n-1}$.

Notice that w acts only on the first n-th coordinates, in other terms:

$$-k[\alpha_n]_w = -k\Big[(-1, 0,\ \dots\ , 0, 1)\Big]_w = -k(0, .., -1, .., 1) = (0, .., k, .., -k),$$

where k is in one of the first n-th positions.

So, $\Big(h\alpha_n - k[\alpha_n]_w, 1\Big) = \Big((-h, .., k, .., h-k), 1\Big) \in \mathbb{Z}_{n-1}$. That is $h = k$.

For the converse, suppose that $x_n \in y_n W(A_{n-1})$ and $h = k$. So $y = x_n u \alpha_n^k z_{n-1}$, for some u in $W(A_{n-1})$. Thus $y = x_n [\alpha_n^k]_u u z_{n-1}$.

As we have shown above, since u is in $W(A_{n-1})$, we get $[\alpha_n^k]_u = \alpha_n^k z''_{n-1}$ for some $z''_{n-1} \in \mathbb{Z}_{n-1}$.

Hence, $yW(\tilde{A_{n-1}}) = x_n \alpha_n^k z''_{n-1} u z_{n-1} W(\tilde{A_{n-1}})$,

so that $yW(\tilde{A_{n-1}}) = xW(\tilde{A_{n-1}})$.

Thus, we have $x \in yW(\tilde{A_{n-1}}) \Leftrightarrow x_n \in y_n W(A_{n-1})$ and $h = k$.

This is equivalent to $L_n = \Big\{\sigma_r..\sigma_n \alpha_n^k;\ (k \in \mathbb{Z}), (1 \le r \le n+1)\Big\}$.

□

Now we treat the double classes of $W(\tilde{A_{n-1}})$ in $W(\tilde{A}_n)$. We start by the following lemma.

Lemma 3.3.6. *Let x, y be any two elements in $W(\tilde{A}_n)$. Suppose that $x = x_n\alpha_n^k z_{n-1}$, $y = y_n\alpha_n^h\zeta_{n-1}$ as above. Then:*

$$[x \in W(\tilde{A_{n-1}})yW(\tilde{A_{n-1}}) \Rightarrow x_n \in W(A_{n-1})y_nW(A_{n-1})].$$

Proof. $x \in W(\tilde{A_{n-1}})yW(\tilde{A_{n-1}})$ gives that:

$$W(\tilde{A_{n-1}})x_n\alpha_n^kW(\tilde{A_{n-1}}) = W(\tilde{A_{n-1}})y_n\alpha_n^hW(\tilde{A_{n-1}}).$$

Hence, there exists w in $W(A_{n-1})$, and λ in $\mathbb{Z}_{n-1}$, such that:

$$\lambda x_n\alpha_n^kW(\tilde{A_{n-1}}) = wy_n\alpha_n^hW(\tilde{A_{n-1}}).$$

Now we consider λx_n, which is equal to $x_n[\lambda]_{x_n^{-1}}$. Thus $\lambda x_n\alpha_n^kW(\tilde{A_{n-1}})$ is equal to $x_n\alpha_n^k[\lambda]_{x_n^{-1}}W(\tilde{A_{n-1}})$. So we have:

$$x_n\alpha_n^k[\lambda]_{x_n^{-1}}W(\tilde{A_{n-1}}) = wy_n\alpha_n^hW(\tilde{A_{n-1}}).$$

In other terms, the two elements $\underbrace{x_n}_{W(A_n)}\alpha_n^k[\lambda]_{x_n^{-1}}$ and $\underbrace{wy_n}_{W(A_n)}\alpha_n^h$ belong to the same left class, hence, by the last proposition there exists some $\grave{w}$ in $W(A_{n-1})$, such that $x_n = wy_n\grave{w}$. That is: $x_n \in A_{n-1}y_nA_{n-1}$.

□

Now we take $x = x_n\alpha_n^kz_{n-1}$ and $y = y_n\alpha_n^h\zeta_{n-1}$ to be any two elements in $\tilde{A}_n$ as above. Suppose that they are in the same double class. By lemma 3.3.6 we have two, and only two cases to be treated:

(1) x_n and y_n are both in $W(A_{n-1})$.

(2) $x_n = u\sigma_n..\sigma_i$, and $y_n = v\sigma_n..\sigma_i$, for v, u in $W(A_{n-1})$, and $1 \leq i \leq n$.

Suppose that we are in case (1).

The fact that $x \in W(\tilde{A_{n-1}})yW(\tilde{A_{n-1}})$ gives $W(\tilde{A_{n-1}})\alpha_n^kW(\tilde{A_{n-1}}) = W(\tilde{A_{n-1}})\alpha_n^hW(\tilde{A_{n-1}})$. Thus, there exists w in $W(A_{n-1})$, and η $\mathbb{Z}_{n-1}$, such that $\eta\alpha_n^kW(\tilde{A_{n-1}}) = w\alpha_n^hW(\tilde{A_{n-1}})$. That is: $\alpha_n^kW(\tilde{A_{n-1}}) = w\alpha_n^hW(\tilde{A_{n-1}})$. By proposition 3.3.5 we have $h = k$. In other terms, when we are in case (1) the double classes are determined by $k \in \mathbb{Z}$.

Suppose that we are in case (2).

The fact that $x \in W(\tilde{A_{n-1}})yW(\tilde{A_{n-1}})$ gives

$W(\tilde{A_{n-1}})x_n\alpha_n^kW(\tilde{A_{n-1}}) = W(\tilde{A_{n-1}})y_n\alpha_n^hW(A_n\tilde{-}1)$. In other words:

$$W(\tilde{A_{n-1}})u\sigma_n\ ...\ \sigma_i\alpha_n^kW(\tilde{A_{n-1}}) = W(\tilde{A_{n-1}})v\sigma_n\ ...\ \sigma_i\alpha_n^h\tilde{A_{n-1}}).$$

But $\sigma_{n-1}\ ...\ \sigma_i\alpha_n^k = \alpha_n^k z''_{n-1}\sigma_{n-1}\ ...\ \sigma_i,$ for some z''_{n-1} in $\mathbb{Z}_{n-1}$. The same holds for α_n^h.

Hence, $x \in W(\tilde{A_{n-1}})yW(\tilde{A_{n-1}})$ gives that:

$$W(\tilde{A_{n-1}})\sigma_n\alpha_n^kW(\tilde{A_{n-1}}) = W(\tilde{A_{n-1}})\sigma_n\alpha_n^hW(\tilde{A_{n-1}}),$$

which is equivalent to $W(\tilde{A_{n-1}})\alpha_{n-1}^k\sigma_nW(\tilde{A_{n-1}}) = W(\tilde{A_{n-1}})\alpha_{n-1}^h\sigma_nW(\tilde{A_{n-1}})$.

Finally we see that in case (2) $x \in W(\tilde{A_{n-1}})yW(\tilde{A_{n-1}})$ implies that

$$W(\tilde{A_{n-1}})\sigma_nW(\tilde{A_{n-1}}) = W(\tilde{A_{n-1}})\sigma_nW(\tilde{A_{n-1}}).$$

This is always verified, in other words when we are in case (2) there is a unique double class presented by σ_n.

Corollary 3.3.7. *Let D_n be the set of distinguished double coset representatives of $W(\tilde{A_{n-1}})$ in $W(\tilde{A}_n)$. Then*

$$D_n = \left\{\alpha_n^k, \sigma_n;\ (k \in \mathbb{Z})\right\}.$$

3.4 Fully commutative elements

In the first part of this section, we give some general definitions. In the second part, we list our general result about the affine fully commutative elements. No details will be given about fully commutative elements in Coxeter groups of types other than the group $W(\tilde{A}_n)$. The concept of fully commutative elements is central in the theory of Coxeter groups, rather then being a base point of the (T-L) algebras theory, about which we are talking details, in the following sections.

Let (W, S) be a Coxeter system with associated Dynkin Diagram Γ. Let $w \in W$. We know that from a given reduced expression of w we can arrive to any other reduced expression only by applying braid relations. Among these relations there are commutation relations corresponding to the non-neighbors (precisely, t and s with $m_{st} = 2$).

Definition 3.4.1. *Elements for which any reduced expression can be arrived to starting from any other only by applying commutation relations are called fully commutative elements. Usually we denote the set of fully commutative element by W^c.*

Remark 3.4.2. * *Suppose that (W, S) is such that any two elements in S are conjugate in W, in this case fully commutative elements have some additional elegant properties, for example we can reformulate the definition as follows.*

Proposition 3.4.3. *Let (W, S) be such that any two elements in S are conjugate in W. Let $w \in W$. Then w is fully commutative if and only if every s in $Supp(w)$ occurs the same number of times in any reduced expression of w.*

Hence, in this case, in a fully commutative element w, we can talk about the multiplicity of a simple reflection in $Supp(w)$. That is if s is in $Supp(w)$, we call the multiplicity of s in w the number of times s appears in a (hence every) reduced expression of w. The center of our interest in this work is fully commutative elements in $\tilde{A}$-type Coxeter groups, which is an example of Coxeter groups in which any two elements in S are conjugate.

3.4.1 Classification of $W^c(\tilde{A}_n)$: a normal form

This subsection is to be viewed as of the proof of the following theorem.

Theorem 3.4.4. *Let $2 \leq n$. Let $w \in W(\tilde{A}_n)$ be a fully commutative, such that $a_{n+1} \in supp(w)$. Then, there exists a unique reduced expression of w, of the following form:*

$$w = \sigma_{i_1}..\sigma_2\sigma_1\sigma_{r_1}..\sigma_{n-1}\sigma_n a_{n+1}\sigma_{i_2}..\sigma_2\sigma_1\sigma_{r_2}..\sigma_{n-1}\sigma_n a_{n+1} \ \ldots \ \sigma_{i_p}..\sigma_2\sigma_1\sigma_{r_p}..\sigma_{n-1}\sigma_n$$
$$(a_{n+1}\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}\sigma_n)^k u.$$

Where: $0 \leq i_1 < i_2 \ \ldots \ < i_p < r_p < r_2 \ \ldots \ < r_1 \leq n+1$, $r_p - i_p \geq 2$, $i_p < j \leq r_p - 1$, $i_1 \leq n$ and $0 \leq k$.

We have two possible forms for u:

- *If $k = 0$, then:*
$$u = a_{n+1}\sigma_{l_1}..\sigma_{g_1}\sigma_{l_2}..\sigma_{g_2} \ \ldots \ \sigma_{l_t}..\sigma_{g_t},$$
where $1 \leq l_1 < l_2.. < l_t \leq n$, $1 \leq g_1 < g_2.. < g_t \leq n$ and $g_k \leq l_k$, for any $1 \leq k \leq n$. With $i_p < l_1$ and $g_t < r_p$.

- *If* $k \geq 1$*, then:*

$$u = a_{n+1}\sigma_j \ ... \ \sigma_{d_1}\sigma_{j+1} \ ... \ \sigma_{d_2}\sigma_{j+2} \ ... \ \sigma_{d_3} \ ... \ \sigma_{j+z} \ ... \ \sigma_{d_{z+1}},$$

where $d_1 < d_2 \ ... \ < d_{z+1}$ *and* $j + c \geq d_{c+1}$ *for* $0 \leq c \leq z$*.*

Consider the group $W(A_n)$. We set $W^c(A_n)$ to be the set of fully commutative elements, its cardinality is the Catalan number $\frac{1}{n+2}\binom{2(n+1)}{n+1}$. However, one can prove easily by induction on n $\big($considering right classes of $W(A_{n-1})$ in $W(A_n)\big)$ the following proposition.

Proposition 3.4.5. *Let* u *be any commutative element in* $W(A_n)$*. Then there is a unique reduced expression of* u *of the form*

$$u = \sigma_{i_1}..\sigma_{j_1}\sigma_{i_2}..\sigma_{j_2} \ ... \ \sigma_{i_p}..\sigma_{j_p},$$

where $1 \leq i_1 < i_2.. < i_p \leq n$*,* $1 \leq j_1 < j_2.. < j_p \leq n$ *and* $j_k \leq i_k$ *for every* $1 \leq k \leq n$*.*

Notice that if σ_n belongs to *supp*(u), then σ_n will certainly appear only once, and it is to be equal to σ_{i_p}. Similarly for σ_1: if it belongs to *supp*(u), then σ_1 will certainly appear only once, and it is equal to σ_{j_1}.

Definition 3.4.6. *An element* u *in* $W^c(A_n)$ *is to be called full if and only if both* σ_n *and* σ_1 *belong to Supp*(w)*. In this case* u *has a reduced expression of the form:*

$$u = \sigma_{i_1}..\sigma_1\sigma_{i_2}..\sigma_{j_2} \ ... \ \sigma_n..\sigma_{j_p},$$

where $1 \leq i_1 < i_2.. < i_{p-1} \leq n$*,* $1 \leq j_2.. < j_p \leq n$ *and* $j_k \leq i_k$ *for every* $1 \leq k \leq n$*.*

Definition 3.4.7. *Suppose that* u *is full, i.e.,* $u = \sigma_{i_1}..\sigma_1\sigma_{i_2}..\sigma_{j_2} \ ... \ \sigma_n..\sigma_{j_p}$*. We say that* σ_n *is on the left (in* u*), if and only if* $u = \sigma_n \ ... \ \sigma_2\sigma_1$*. In all other cases we say that* σ_n *is on the right.*

Definition 3.4.8. *We define the affine length of* u *in* $W^c(A_n)$ *to be: the multiplicity of* a_{n+1} *in Supp*(u)*. We denoted it by* $L(u)$*.*

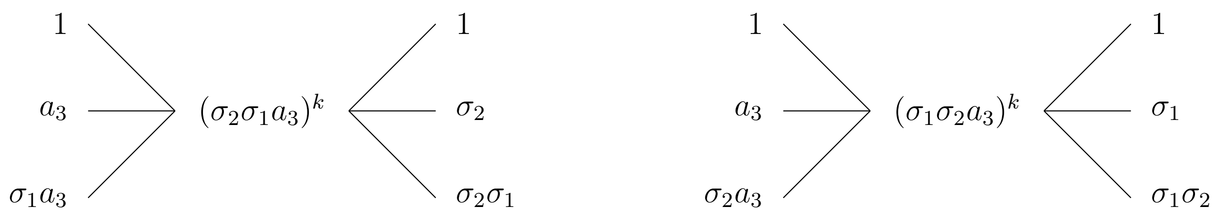
Suppose that w is a fully commutative element in $W(\tilde{A}_n)$. Clearly $L(w) = 0$ expresses the case where a_{n+1} is not in *supp*(w), in other terms w is a fully commutative element in $W(A_n)$. Suppose that $L(w) = m$ where m is positive. Any reduced expression of w is of the form:

$$w = u_1 a_{n+1} u_2 a_{n+1} \ \dots \ u_m a_{n+1} u_{m+1},$$

where u_i is in $W^c(A_n)$, for $1 \leq i \leq m+1$. Moreover, suppose that $L(w) \geq 2$. Then u_i must be full for $2 \leq u_i \leq m$, otherwise w is not fully commutative.

Before treating the general case, we classify fully commutative elements of $W(\tilde{A}_2)$. This gives an idea about the general proof, in its most simple form.

Theorem 3.4.9. *Let w be in $W^c(\tilde{A}_2)$. Suppose that $0 \leq L(w)$. Then there exists $0 \leq k$, such that w has one and only one of the following forms:*



Proof. As we saw above $w = u_1 a_3 u_2 a_3 \ \dots \ u_m a_3 u_{m+1}$, where u_i is in $W(A_2)$. If $L(w)$ is 1 or 2 it is clear that we can get it from the tree formulas above. Suppose that $2 \leq L(w)$. Hence u_i is full for $2 \leq u_i \leq m$. In particular u_2 is full. Actually there are not many choices for u_2, since the only full elements in $W(A_2)$ are $\sigma_1\sigma_2$ and $\sigma_2\sigma_1$. The first possibility is that $u_2 = \sigma_1\sigma_2$. Now being a full element, u_3 is definitely equal to $\sigma_1\sigma_2$, otherwise we would have the, in w, the following subword $u_1 a_{n+1}\sigma_1 \underbrace{\sigma_2 a_3 \sigma_2}\sigma_1$. This is not possible since w is fully commutative, thus $u_3 = u_2 = \sigma_1\sigma_2$. The same holds for every u_i for $i \leq m$, i.e., if u_2 is equal to $\sigma_1\sigma_2$ then $w = u_1 a_{n+1}(\sigma_1\sigma_2 a_3)^{m-1} u_{m+1}$.

It is clear that u_1 is in $W(A_2)$, and does not end with σ_1, hence u_1 is equal to σ_2 or 1. In the same way, we see that u_{m+1} is in $W^c(A_2)$, it cannot ends with σ_2, so u_{m+1} is equal to σ_1,$\sigma_1\sigma_2$ or 1. In other terms, if u_2 is equal to $\sigma_1\sigma_2$ we get the second tree.

Now suppose that $u_2 = \sigma_2\sigma_1$, then $w = u_1 a_3 (\sigma_2\sigma_1 a_3)^{m-1} u_{m+1}$. With a similar discussion about the first choice of u_2, we see that when $u_2 = \sigma_2\sigma_1$ we get the first tree.

∎

In order to simplify, we suppose now that $3 \leq n$ (Although many propositions in what follows are valid in $W(\tilde{A}_2)$).

Remark 3.4.10. *Let u be a full element : $u = \sigma_{i_1}..\sigma_1\sigma_{i_2}..\sigma_{j_2} \dots \sigma_n..\sigma_{j_p}$. Assume that σ_n is on the right in u, hence, by pushing σ_n to the left we see easily that*

$$u = \sigma_i..\sigma_2\sigma_1\sigma_r \dots \sigma_{n-1}\sigma_n x,$$

where $1 \leq i \leq n-1$, $1 \leq r \leq n$ and $i < r$. While $supp(x) \subseteq \{\sigma_2, \sigma_3..\sigma_{n-1}\}$ if x is not 1.

Lemma 3.4.11. *Let w be in $W^c(\tilde{A}_n)$ such that $2 \leq L(w)$. Say:*

$$w = u_1 a_{n+1} u_2 a_{n+1} \dots u_m a_{n+1} u_{m+1}.$$

Assume that σ_n is on the right in u_h, for $2 \leq h \leq m$. Then w has one of the three following forms:

$$\begin{aligned} w_1 = {} & u_1 a_{n+1}\sigma_{i_1}..\sigma_2\sigma_1\sigma_{r_1}..\sigma_{n-1}\sigma_n a_{n+1}\sigma_{i_2}..\sigma_2\sigma_1\sigma_{r_2}..\sigma_{n-1}\sigma_n a_{n+1} \dots \sigma_{i_p}..\sigma_2\sigma_1\sigma_{r_p}..\sigma_{n-1}\sigma_n \\ & (a_{n+1}\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}\sigma_n)^{m-(1+p)} \\ & a_{n+1}\sigma_j..\sigma_{d_1}\sigma_{j+1}..\sigma_{d_2}\sigma_{j+2}..\sigma_{d_3} \dots \sigma_{j+z}..\sigma_{d_{z+1}} \end{aligned}$$

Where $i_1 < i_2 \dots < i_p < r_p < r_2 \dots < r_1 \leq n$, $r_p - i_p \geq 3$ and $p < n/2$.

With $i_p < j$ and $j+1 < r_p$.

While $d_1 < d_2 \dots < d_{z+1}$ and $j + c \geq d_{c+1}$, for $0 \leq c \leq z$.

$$\begin{aligned} w_2 = {} & u_1 a_{n+1}\sigma_{i_1}..\sigma_2\sigma_1\sigma_{r_1}..\sigma_{n-1}\sigma_n a_{n+1}\sigma_{i_2}..\sigma_2\sigma_1\sigma_{r_2}..\sigma_{n-1}\sigma_n a_{n+1} \dots \sigma_{i_p}..\sigma_2\sigma_1\sigma_{r_p}..\sigma_{n-1}\sigma_n \\ & a_{n+1}\sigma_j..\sigma_2\sigma_1\sigma_{j+2}..\sigma_{n-1}\sigma_n \\ & (a_{n+1}\sigma_{j+1}..\sigma_2\sigma_1\sigma_{j+2}..\sigma_{n-1}\sigma_n)^{m-(p+2)} \\ & a_{n+1}\sigma_{j+1}..\sigma_{d_1}\sigma_{j+2}..\sigma_{d_2}\sigma_{j+3}..\sigma_{d_3} \dots \sigma_{j+z}..\sigma_{d_z} \end{aligned}$$

Where $i_1 < i_2 \dots < i_p < r_p < r_2 \dots r_1 \leq n$, $r_p - i_p \geq 4$, and $p < n/2$.

With $i_p < j$ and $j+2 < r_p$.

While $d_1 < d_2 \dots < d_z$ and $j + c \geq d_{c+1}$ for $0 \leq c \leq z$.

$$\begin{aligned} w_3 = {} & u_1 a_{n+1}\sigma_{i_1}..\sigma_2\sigma_1\sigma_{r_1}..\sigma_{n-1}\sigma_n a_{n+1}\sigma_{i_2}..\sigma_2\sigma_1\sigma_{r_2}..\sigma_{n-1}\sigma_n a_{n+1} \dots \sigma_{i_p}..\sigma_2\sigma_1\sigma_{r_p} \\ & a_{n+1}\sigma_{l_1}..\sigma_{g_1}\sigma_{l_2}..\sigma_{g_2} \dots \sigma_{l_t}..\sigma_{g_t}. \end{aligned}$$

Where $i_1 < i_2 \ ... \ < i_p < r_p < r_2 \ ... \ r_1 \leq n$, $r_p - i_p \geq 3$ *and* $p < n/2$.

With $1 \leq l_1 < l_2.. < l_t \leq n$ *and* $1 \leq g_1 < g_2.. < g_t \leq n$.

While $i_p < l_1$, $g_t < r_p$ *and* $g_k \leq l_k$ *for any* $1 \leq k \leq n$.

Proof. Before starting with the details of the proof, we call the reader's attention to the fact that our assumption that σ_n is on the right in u_h for $2 \leq h \leq m$ is legitimate, since we know that these u_h are full by the discussion above. Using the discussion above we can write:

$$u_{h-1} = \sigma_{i_h}..\sigma_2\sigma_1\sigma_{r_h} \ ... \ \sigma_{n-1}\sigma_n x_h, \text{ for } 3 \leq h \leq m+1.$$

As above $1 \leq i_h \leq n-1$, $1 \leq r_h \leq n$ and $i_h < r_h$, with $Supp(x_h) \subseteq \{\sigma_2, \sigma_3..\sigma_{n-1},\}$. Since a_{n+1} commutes with x_h for all h, we can write $x_i a_{n+1} u_{i+1}$ as $a_{n+1}u'_{i+1}$ with u'_{i+1} full, in which σ_n is on the right. Applying this inductively, we can write w as follows:

$$w = u_1 a_{n+1}\sigma_{i_1}..\sigma_2\sigma_1\sigma_{r_1}..\sigma_{n-1}\sigma_n a_{n+1}\sigma_{i_2}..\sigma_2\sigma_1\sigma_{r_2}..\sigma_{n-1}\sigma_n a_{n+1} \ ... \ \sigma_{i_{m-1}}..\sigma_2\sigma_1\sigma_{r_{m-1}} \ ... \\ ... \ \sigma_{n-1}\sigma_n a_{n+1} u_{m+1},$$

with u_1, u_{m+1}, i_h and r_h as above. Now we have 3 main cases to consider:

(1) $r_1 - i_1 = 1$, i.e., $r_1 = i_1 + 1$.

In this case we do not have many choices for the full elements on the right of u_2: we have one and only one choice, $i_h = i_1$ for all $h \leq m-1$. Thus $j = i_1$. We have:

$$w = u_1(a_{n+1}\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}\sigma_n)^{m-1}a_{n+1}u_{m+1}.$$

Here we see that u_{m+1} is a fully commutative element, which need not to be full, yet this element cannot have a reduced expression starting by any simple reflection in $W(A_n)$ but σ_{i_1}. If $u_{m+1} \neq 1$, we can thus, express it as follows:

$$u_{m+1} = \sigma_j..\sigma_{d_1}\sigma_{j+1}..\sigma_{d_2}\sigma_{j+2}..\sigma_{d_3} \ ... \ \sigma_{j+z}..\sigma_{d_{z+1}},$$

$$\text{where } d_1 < d_2 \ ... \ < d_{z+1} \text{ and } j + c \geq d_{c+1} \text{ for } 0 \leq c \leq z.$$

(2) $r_1 - i_1 = 2$, i.e., $r_1 = i_1 + 2$.

In this case we have, as well, only one choice for the full element on the right of u_2, namely (we set $i_1 = j$):

$$w = u_1 a_{n+1}\sigma_j..\sigma_2\sigma_1\sigma_{j+2}..\sigma_{n-1}\sigma_n(a_{n+1}\sigma_{j+1}..\sigma_2\sigma_1\sigma_{j+2}..\sigma_{n-1}\sigma_n)^{m-2}a_{n+1}u_{m+1},$$

with conditions on u_{m+1} analogous to those of case (1), that is:

$$u_{m+1} = \sigma_{j+1}..\sigma_{d_1}\sigma_{j+2}..\sigma_{d_2}\sigma_{i_1+3}..\sigma_{d_3}..\sigma_{j+z}..\sigma_{d_z},$$

where $d_1 < d_2 \ ... \ < d_z$ and $j + c \geq d_c$ for $1 \leq c \leq z$.

(3) $r_1 - i_1 > 2$.

$$\text{Say } w = u_1 a_{n+1}\sigma_{i_1}..\sigma_2\sigma_1\sigma_{r_1}..\sigma_{n-1}\sigma_n a_{n+1}\sigma_{i_2}..\sigma_2\sigma_1\sigma_{r_2}..\sigma_{n-1}\sigma_n$$
$$a_{n+1} \ ... \ \sigma_{i_{m-1}}..\sigma_2\sigma_1\sigma_{r_{m-1}} \ ... \ \sigma_{n-1}\sigma_n a_{n+1}u_{m+1}.$$

We see that we have to choose r_2 and i_2 such that $i_1 < i_2 < r_2 < r_1$. Hence, after a finite number of steps, we will face one of the cases (1) or (2). Thus we have one of the next forms:

(1') This is the case related to (1), i.e., we have:

$$w = u_1 a_{n+1}\sigma_{i_1}..\sigma_2\sigma_1\sigma_{r_1}..\sigma_{n-1}\sigma_n a_{n+1}\sigma_{i_2}..\sigma_2\sigma_1\sigma_{r_2}..\sigma_{n-1}\sigma_n a_{n+1} \ ... \ \sigma_{i_p}..\sigma_2\sigma_1\sigma_{r_p}..\sigma_{n-1}\sigma_n$$
$$(a_{n+1}\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}\sigma_n)^{m-(1+p)}a_{n+1}u_{m+1}.$$

Here $i_1 < i_2 \ ... \ < i_p < r_p < r_2 \ ... \ < r_1 \leq n$ and $r_p - i_p \geq 3$. We have necessarily $p < n/2$, while u_{m+1} is as in case (1).

(2') This case is related to (2), i.e., we have:

$$w = u_1 a_{n+1}\sigma_{i_1}..\sigma_2\sigma_1\sigma_{r_1}..\sigma_{n-1}\sigma_n a_{n+1}\sigma_{i_2}..\sigma_2\sigma_1\sigma_{r_2}..\sigma_{n-1}\sigma_n a_{n+1} \ ... \ \sigma_{i_p}..\sigma_2\sigma_1\sigma_{r_p}..\sigma_{n-1}\sigma_n$$
$$a_{n+1}\sigma_j..\sigma_2\sigma_1\sigma_{j+2}..\sigma_{n-1}\sigma_n(a_{n+1}\sigma_{j+1}..\sigma_2\sigma_1\sigma_{j+2}..\sigma_{n-1}\sigma_n)^{m-(p+2)}a_{n+1}u_{m+1}.$$

Here $i_1 < i_2 \ ... \ < i_p < r_p < r_2 \ ... \ < r_1 \leq n$ and $r_p - i_p \geq 4$. We have necessarily $p < n/2$, while u_{m+1} is as in case (2).

(3') This case is related to some "short" elements (with respect to L):

suppose that we stopped picking pairs (i, r) before having a difference of 1 or 2 between them, hence:

$$w = u_1 a_{n+1}\sigma_{i_1}..\sigma_2\sigma_1\sigma_{r_1}..\sigma_{n-1}\sigma_n a_{n+1}\sigma_{i_2}..\sigma_2\sigma_1\sigma_{r_2}..\sigma_{n-1}\sigma_n a_{n+1} \ ... \ \sigma_{i_p}..\sigma_2\sigma_1\sigma_{r_p}a_{n+1}u_{m+1},$$

with $i_1 < i_2 \ ... \ < i_p < r_p < r_2 \ ... \ < r_1 \leq n$ and $r_p - i_p \geq 3$. We have necessarily $p < n/2$.

In this case, the choice of u_{m+1} is much more complicated than in the other two cases. It has the form:

$$\sigma_{l_1}..\sigma_{g_1}\sigma_{l_2}..\sigma_{g_2}\ \dots\ \sigma_{l_t}..\sigma_{g_t},$$

where $1 \leq l_1 < l_2.. < l_t \leq n$, $1 \leq g_1 < g_2.. < g_t \leq n$ and $g_k \leq l_k$ for any $1 \leq k \leq n$. And in addition we have $i_p < l_1$ and $g_t < r_p$.

Later on, We will be back to handle the possible forms of u_{m+1}.

□

Definition 3.4.12. *In elements of type w_1, the following element is called the short block:*

$$a_{n+1}\sigma_{i_1}..\sigma_2\sigma_1\sigma_{r_1}..\sigma_{n-1}\sigma_n a_{n+1}\sigma_{i_2}..\sigma_2\sigma_1\sigma_{r_2}..\sigma_{n-1}\sigma_n a_{n+1}\ \dots\ \sigma_{i_p}..\sigma_2\sigma_1\sigma_{r_p}..\sigma_{n-1}\sigma_n.$$

We call $(a_{n+1}\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}\sigma_n)^{m-(1+p)}$ the convergent block of w_1.

We call $a_{n+1}u_{m+1}$ the residue block of w_1.

Hence we can write $w_1 = u_1$. short block. convergent block. residue block. (We do the same thing for elements of type w_2, in which for example, the convergent block is $(a_{n+1}\sigma_{j+1}..\sigma_2\sigma_1\sigma_{j+2}..\sigma_{n-1}\sigma_n)^{m-(p+2)}$).

Definition 3.4.13. *An element of the last two types is to be called short, if and only if its convergent block is equal to 1.*

Remark 3.4.14. * *It is easy to see that w_1 and w_2 could be unified in the following form:*

$$\begin{aligned} w_1 = u_1 a_{n+1}\sigma_{i_1}..\sigma_2\sigma_1\sigma_{r_1}..\sigma_{n-1}\sigma_n a_{n+1}\sigma_{i_2}..\sigma_2\sigma_1\sigma_{r_2}..\sigma_{n-1}\sigma_n a_{n+1}\ \dots\ \sigma_{i_p}..\sigma_2\sigma_1\sigma_{r_p}..\sigma_{n-1}\sigma_n \\ \Big(a_{n+1}\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}\sigma_n\Big)^{m-\Big(1+L(\text{the short block})\Big)} \\ a_{n+1}\sigma_j..\sigma_{d_1}\sigma_{j+1}..\sigma_{d_2}\sigma_{i_1+2}..\sigma_{d_3}\ \dots\ \sigma_{i_1+z}..\sigma_{d_{z+1}}, \end{aligned}$$

> *where: $i_1 < i_2\ \dots\ < i_p < r_p < r_2\ \dots\ < r_1 \leq n$, $r_p - i_p \geq 2$, and p necessarily lesser than $n/2$.*
>
> *while $i_p < j$, $j+1 < r_p$, $d_1 < d_2\ \dots\ < d_{z+1}$ and $j + c \geq d_{c+1}$ for $0 \leq c \leq z$.*

Nevertheless, for the moment, we will go on keeping looking at them as two different forms .

We see that the set of short elements is of finite cardinal, because of the fact that the affine length L of such elements is bounded. Special cases of the last lemma, which comes from the 3 types above when $u_{m+1} = 1$, are included in the general formula.

Now we classify the elements of $W^c(\tilde{A}_n)$ with $n \geq 3$.

Consider an arbitrary w in $W^c(\tilde{A}_n)$ with $L(w) \geq 2$, written as:

$$w = u_1 a_{n+1} u_2 a_{n+1} \; ... \; u_m a_{n+1} u_{m+1}.$$

We start the classification, depending on the choice of u_1 which can have one, and only one of the following forms:

- (a) u_1 is full, with σ_n on the left.
- (b) u_1 is full, with σ_n on the left.
- (c) σ_n belongs to $supp(u_1)$ and σ_1 does not.
- (d) σ_1 belongs to $supp(u_1)$ and σ_n does not.
- (e) $u_1 = 1$.

Suppose that we are in case (a).

We have $u_1 = \sigma_n \sigma_{n-1} \; ... \; \sigma_2 \sigma_1$. In this case there is only one choice for the full elements u_i with $2 \leq i \leq m$, which is to be equal to u_1, hence $w = (\sigma_n \sigma_{n-1} \; ... \; \sigma_2 \sigma_1 a_{n+1})^m u_{m+1}$. Here u_{m+1} is either 1 or $\sigma_n \sigma_{n-1} \; ... \; \sigma_j$, thus we have two possible types:

$$x_1 = (\sigma_n \sigma_{n-1} \; ... \; \sigma_2 \sigma_1 a_{n+1})^m \sigma_n \sigma_{n-1} \; ... \; \sigma_i, \text{ for } 1 \leq i \leq n.$$
$$x_2 = (\sigma_n \sigma_{n-1} \; ... \; \sigma_2 \sigma_1 a_{n+1})^m.$$

Suppose that we are in case (b).

Set $u_1 := \sigma_{i_0} .. \sigma_2 \sigma_1 \sigma_{r_0} \; ... \; \sigma_{n-1} \sigma_n x_0$. It is clear that u_i, for $2 \leq i$, cannot be equal to $\sigma_n \; ... \; \sigma_1$, hence all the full elements u_i, for $2 \leq i \leq m$, have σ_n on the right. Here we can use the same discussion as in lemma.2.4.9. We arrive to the possible types, by replacing $u_1 a_{n+1}$ (in which w starts) by 1. Thus we have three possible types (modulo maybe a shift of indexes to the left):

$$\begin{aligned} x_3 = {} & \sigma_{i_1} .. \sigma_2 \sigma_1 \sigma_{r_1} .. \sigma_{n-1} \sigma_n a_{n+1} \sigma_{i_2} .. \sigma_2 \sigma_1 \sigma_{r_2} .. \sigma_{n-1} \sigma_n a_{n+1} \; ... \; \sigma_{i_p} .. \sigma_2 \sigma_1 \sigma_{r_p} .. \sigma_{n-1} \sigma_n \\ & (a_{n+1} \sigma_j .. \sigma_2 \sigma_1 \sigma_{j+1} .. \sigma_{n-1} \sigma_n)^{m-p} \\ & a_{n+1} \sigma_j .. \sigma_{d_1} \sigma_{j+1} .. \sigma_{d_2} \sigma_{j+2} .. \sigma_{d_3} \; ... \; \sigma_{j+z} .. \sigma_{d_{z+1}}. \end{aligned}$$

Where $i_1 < i_2 \ldots < i_p < r_p < r_2 \ldots < r_1 \leq n$, $r_p - i_p \geq 3$ and $p < n/2$.

With $i_p < j$ and $j + 1 < r_p$.

While $d_1 < d_2 \ldots < d_{z+1}$ and $j + c \geq d_{c+1}$ for $0 \leq c \leq z$.

$$x_4 = \sigma_{i_1}..\sigma_2\sigma_1\sigma_{r_1}..\sigma_{n-1}\sigma_n a_{n+1}\sigma_{i_2}..\sigma_2\sigma_1\sigma_{r_2}..\sigma_{n-1}\sigma_n a_{n+1} \ldots \sigma_{i_p}..\sigma_2\sigma_1\sigma_{r_p}..\sigma_{n-1}\sigma_n$$
$$a_{n+1}\sigma_j..\sigma_2\sigma_1\sigma_{j+2}..\sigma_{n-1}\sigma_n(a_{n+1}\sigma_{j+1}..\sigma_2\sigma_1\sigma_{j+2}..\sigma_{n-1}\sigma_n)^{m-(p+1)}$$
$$a_{n+1}\sigma_{j+1}..\sigma_{d_1}\sigma_{j+2}..\sigma_{d_2}\sigma_{j+3}..\sigma_{d_3} \ldots \sigma_{j+z}..\sigma_{d_z}.$$

Where $i_1 < i_2 \ldots < i_p < r_p < r_2 \ldots < r_1 \leq n$, $r_p - i_p \geq 4$ and $p < n/2$.

With $i_p < j$ and $j + 2 < r_p$.

While $d_1 < d_2 \ldots < d_z$ and $i_1 + c \geq d_c$ for $1 \leq c \leq z$.

$$x_5 = \sigma_{i_1}..\sigma_2\sigma_1\sigma_{r_1}..\sigma_{n-1}\sigma_n a_{n+1}\sigma_{i_2}..\sigma_2\sigma_1\sigma_{r_2}..\sigma_{n-1}\sigma_n a_{n+1} \ldots \sigma_{i_p}..\sigma_2\sigma_1\sigma_{r_p}$$
$$a_{n+1}\sigma_{l_1}..\sigma_{g_1}\sigma_{l_2}..\sigma_{g_2} \ldots \sigma_{l_t}..\sigma_{g_t}.$$

Where $i_1 < i_2 \ldots < i_p < r_p < r_2 \ldots < r_1 \leq n$, $r_p - i_p \geq 3$ and $p < n/2$.

With $1 \leq l_1 < l_2.. < l_t \leq n$ and $1 \leq g_1 < g_2.. < g_t \leq n$.

While $i_p < l_1$, $g_t < r_p$ and $g_k \leq l_k$ for any $1 \leq k \leq n$.

Of course, we keep in mind the three special cases x'_3, (resp. x'_4 and x'_5), which are obtained from x_3, (resp. x_4 and x_5) by replacing u_{m+1} by 1.

Suppose that we are in case (c).

Here, u_1 can be written as $u_1 = \sigma_h\sigma_{h+1}..\sigma_{n-1}\sigma_n y$, where $2 \leq h \leq n$, y is in $W(A_{n-1})$ and $\sigma_1 \notin supp(y)$. Hence, we can write w as follows:

$$w = \sigma_h..\sigma_n a_{n+1}u_2 a_{n+1} \ldots u_m a_{n+1}u_{m+1}.$$

We see that u_2 (thus every u_i with $2 \leq i \leq m+1$) cannot start with σ_n. That means each u_i, with $2 \leq i \leq m$, is a full element in which σ_n is on the left. By using the lemma 3.4.11 w is one of the three following elements:

$$x_6 = \sigma_h..\sigma_n a_{n+1}\sigma_{i_1}..\sigma_2\sigma_1\sigma_{r_1}..\sigma_{n-1}\sigma_n a_{n+1}\sigma_{i_2}..\sigma_2\sigma_1\sigma_{r_2}..\sigma_{n-1}\sigma_n a_{n+1} \ldots \sigma_{i_p}..\sigma_2\sigma_1\sigma_{r_p} \ldots \sigma_{n-1}\sigma_n$$
$$(a_{n+1}\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}\sigma_n)^{m-(1+p)}$$
$$a_{n+1}\sigma_j..\sigma_{d_1}\sigma_{j+1}..\sigma_{d_2}\sigma_{j+2}..\sigma_{d_3} \ldots \sigma_{j+z}..\sigma_{d_{z+1}}.$$

Where $i_1 < i_2 \ ... \ < i_p < r_p < r_2 \ ... \ < r_1 \leq n$, $r_p - i_p \geq 3$ and $p < n/2$.

With $i_p < j$, $j+1 < r_p$, $d_1 < d_2 \ ... \ < d_{z+1}$ and $j + c \geq d_{c+1}$ for $0 \leq c \leq z$.

While $i_1 < h$, and if $r_1 - i_1 > 1$, then $h < r_1$.

$$\begin{aligned}x_7 = \ & \sigma_h..\sigma_n a_{n+1}\sigma_{i_1}..\sigma_2\sigma_1\sigma_{r_1}..\sigma_{n-1}\sigma_n a_{n+1}\sigma_{i_2}..\sigma_2\sigma_1\sigma_{r_2}..\sigma_{n-1}\sigma_n a_{n+1} \ ... \ \sigma_{i_p}..\sigma_2\sigma_1\sigma_{r_p} \ ... \ \sigma_{n-1}\sigma_n \\ & a_{n+1}\sigma_{i_1}..\sigma_2\sigma_1\sigma_{i_1+2}..\sigma_{n-1}\sigma_n \\ & (a_{n+1}\sigma_{j+1}..\sigma_2\sigma_1\sigma_{j+2}..\sigma_{n-1}\sigma_n)^{m-(p+2)} \\ & a_{n+1}\sigma_{j+1}..\sigma_{d_1}\sigma_{j+2}..\sigma_{d_2}\sigma_{j+3}..\sigma_{d_3} \ ... \ \sigma_{j+z}..\sigma_{d_z}.\end{aligned}$$

Where $i_1 < i_2 \ ... \ < i_p < r_p < r_2 \ ... \ < r_1 \leq n$, $r_p - i_p \geq 4$ and $p < n/2$.

With $i_p < i_1$, $i_1 + 2 < r_p$, $d_1 < d_2 \ ... \ < d_z$ and $i_1 + c \geq d_c$ for $1 \leq c \leq z$.

While $i_1 < h$, and if $r_1 - i_1 > 1$ then $h < r_1$.

$$\begin{aligned}x_8 = \ & \sigma_h..\sigma_n a_{n+1}\sigma_{i_1}..\sigma_2\sigma_1\sigma_{r_1}..\sigma_{n-1}\sigma_n a_{n+1}\sigma_{i_2}..\sigma_2\sigma_1\sigma_{r_2}..\sigma_{n-1}\sigma_n a_{n+1} \ ... \ \sigma_{i_p}..\sigma_2\sigma_1\sigma_{r_p} \\ & a_{n+1}\sigma_{l_1}..\sigma_{g_1}\sigma_{l_2}..\sigma_{g_2} \ ... \ \sigma_{l_t}..\sigma_{g_t}.\end{aligned}$$

Here $i_1 < i_2 \ ... \ < i_p < r_p < r_2 \ ... \ < r_1 \leq n$, $r_p - i_p \geq 3$ and $p < n/2$.

Where $1 \leq l_1 < l_2.. < l_t \leq n$ and $1 \leq g_1 < g_2.. < g_t \leq n$.

With $g_k \leq l_k$ for any $1 \leq k \leq n$, $i_p < l_1$ and $g_t < r_p$.

While $i_1 < h$, and if $r_1 - i_1 > 1$ then $h < r_1$.

As before we keep in mind the three special cases x'_6, (resp. x'_7 and x'_8), which are obtained from x_6, (resp. x_7 and x_8) by replacing u_{m+1} by 1.

Suppose that we are in case (d).

Here, u_1 can be written $\sigma_h\sigma_{h-1}..\sigma_1 y$, where y is in $W(A_{n-1})$, with $\sigma_1 \notin supp(y)$ and $1 \leq h \leq n-1$. Hence we can suppose that

$$w = \sigma_h..\sigma_1 a_{n+1} u_2 a_{n+1} \ ... \ u_m a_{n+1} u_{m+1}.$$

Here, we have two main choices for u_2. The first one is that σ_n is on the left, then w has the following form:

$x_9 = \sigma_h..\sigma_1 a_{n+1}(\sigma_n\sigma_{n-1} \ ... \ \sigma_2\sigma_1 a_{n+1})^m \sigma_n \ ... \ \sigma_i$. Where $1 \leq h \leq n-1$, and $1 \leq i \leq n$.

The second choice is that u_i, for $2 \leq i \leq n$, has σ_n on the right. As above we have three forms, namely:

$$x_{10} = \sigma_h..\sigma_1 a_{n+1}\sigma_{i_1}..\sigma_2\sigma_1\sigma_{r_1}..\sigma_{n-1}\sigma_n a_{n+1}\sigma_{i_2}..\sigma_2\sigma_1\sigma_{r_2}..\sigma_{n-1}\sigma_n a_{n+1} \ \dots \ \sigma_{i_p}..\sigma_2\sigma_1\sigma_{r_p} \ \dots \ \sigma_{n-1}\sigma_n$$
$$(a_{n+1}\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}\sigma_n)^{m-(1+p)}$$
$$a_{n+1}\sigma_j..\sigma_{d_1}\sigma_{j+1}..\sigma_{d_2}\sigma_{j+2}..\sigma_{d_3} \ \dots \ \sigma_{j+z}..\sigma_{d_{z+1}}.$$

Where $i_1 < i_2 \dots < i_p < r_p < r_2 \dots < r_1 \leq n$ and $r_p - i_p \geq 3$ and $p < n/2$.

With $i_p < i_1$, $i_1 + 1 < r_p$, $d_1 < d_2 \dots < d_{z+1}$ and $i_1 + c \geq d_{c+1}$ for $0 \leq c \leq z$.

While $i_1 < h$, and if $r_1 - i_1 > 1$, then $h < r_1$.

$$x_{11} = \sigma_h..\sigma_1 a_{n+1}\sigma_{i_1}..\sigma_2\sigma_1\sigma_{r_1}..\sigma_{n-1}\sigma_n a_{n+1}\sigma_{i_2}..\sigma_2\sigma_1\sigma_{r_2}..\sigma_{n-1}\sigma_n a_{n+1} \ \dots \ \sigma_{i_p}..\sigma_2\sigma_1\sigma_{r_p} \ \dots \ \sigma_{n-1}\sigma_n$$
$$a_{n+1}\sigma_{i_1}..\sigma_2\sigma_1\sigma_{i_1+2}..\sigma_{n-1}\sigma_n$$
$$(a_{n+1}\sigma_{j+1}..\sigma_2\sigma_1\sigma_{j+2}..\sigma_{n-1}\sigma_n)^{m-(p+2)}$$
$$a_{n+1}\sigma_{j+1}..\sigma_{d_1}\sigma_{j+2}..\sigma_{d_2}\sigma_{j+3}..\sigma_{d_3} \ \dots \ \sigma_{j+z}..\sigma_{d_z}.$$

Where $i_1 < i_2 \dots < i_p < r_p < r_2 \dots < r_1 \leq n$, $r_p - i_p \geq 4$ and $p < n/2$.

With $i_p < i_1$, $i_1 + 2 < r_p$, $d_1 < d_2 \dots < d_z$ and $i_1 + c \geq d_c$ for $1 \leq c \leq z$.

While $i_1 < h$, and if $r_1 - i_1 > 1$, then $h < r_1$.

$$x_{12} = \sigma_h..\sigma_1 a_{n+1}\sigma_{i_1}..\sigma_2\sigma_1\sigma_{r_1}..\sigma_{n-1}\sigma_n a_{n+1}\sigma_{i_2}..\sigma_2\sigma_1\sigma_{r_2}..\sigma_{n-1}\sigma_n a_{n+1} \ \dots \ \sigma_{i_p}..\sigma_2\sigma_1\sigma_{r_p}$$
$$a_{n+1}\sigma_{l_1}..\sigma_{g_1}\sigma_{l_2}..\sigma_{g_2} \ \dots \ \sigma_{l_t}..\sigma_{g_t}.$$

Where $i_1 < i_2 \dots < i_p < r_p < r_2 \dots < r_1 \leq n$, $r_p - i_p \geq 3$ and $p < n/2$.

With $1 \leq l_1 < l_2.. < l_t \leq n$ and $1 \leq g_1 < g_2.. < g_t \leq n$.

While $g_k \leq l_k$, for any $1 \leq k \leq n$, with $i_p < l_1$, $g_t < r_p$ and $i_1 < h < r_1$.

Still, we keep in mind the three special cases x'_9, (resp. x'_{10} and x'_{11}), which are obtained from x_9, (resp. x_{10} and x_{11}) by replacing u_{m+1} by 1.

Suppose that we are in case (e).

This case will be a particular case of the above cases. We use the following notation in $W(\tilde{A}_n)$: $\sigma_0 = \sigma_{n+1} = 1$. With this notation we see that types x_1, x_2 and x_9 could be unified in one form, say c_1.

Moreover, x_3 (resp. x_4 and x_5) can be unified in one form with x_6 (resp. x_7 and x_8), when $i_1 = 0$.

Similarly, x_3 (resp. x_4 and x_5) can be unified in one form with x_{10} (resp. x_{11} and x_{12}), when $r_1 = n + 1$.

From what precedes, we formulate our classification by the following corollary.

Corollary 3.4.15. *Let $3 \leq n$. Let w be in $W^c(\tilde{A}_n)$, such that $2 \leq L(w)$. Then w has one of the following forms:*

$c_1 = \sigma_j \ \dots\ \sigma_2\sigma_1(\sigma_n\sigma_{n-1} \ \dots\ \sigma_2\sigma_1 a_{n+1})^m\sigma_n\sigma_{n-1} \ \dots\ \sigma_i$

Where $1 \leq i \leq n+1$ and $0 \leq j \leq n$.

$$c_2 = \sigma_{i_1}..\sigma_2\sigma_1\sigma_{r_1}..\sigma_{n-1}\sigma_n a_{n+1}\sigma_{i_2}..\sigma_2\sigma_1\sigma_{r_2}..\sigma_{n-1}\sigma_n a_{n+1} \ \dots\ \sigma_{i_p}..\sigma_2\sigma_1\sigma_{r_p} \ \dots\ \sigma_{n-1}\sigma_n$$
$$(a_{n+1}\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}\sigma_n)^{m-(p)}$$
$$a_{n+1}\sigma_j..\sigma_{d_1}\sigma_{j+1}..\sigma_{d_2}\sigma_{j+2}..\sigma_{d_3} \ \dots\ \sigma_{j+z}..\sigma_{d_{z+1}}.$$

Where $0 \leq i_1 < i_2 \ \dots\ < i_p < r_p < r_2 \ \dots\ < r_1 \leq n+1$ and $r_p - i_p \geq 3$.

With $i_p < j$ and $j + 1 < r_p$.

While $d_1 < d_2 \ \dots\ < d_{z+1}$ and $j + c \geq d_{c+1}$ for $0 \leq c \leq z$.

$$c_3 = \sigma_{i_1}..\sigma_2\sigma_1\sigma_{r_1}..\sigma_{n-1}\sigma_n a_{n+1}\sigma_{i_2}..\sigma_2\sigma_1\sigma_{r_2}..\sigma_{n-1}\sigma_n a_{n+1} \ \dots\ \sigma_{i_p}..\sigma_2\sigma_1\sigma_{r_p} \ \dots\ \sigma_{n-1}\sigma_n$$
$$a_{n+1}\sigma_j..\sigma_2\sigma_1\sigma_{j+2}..\sigma_{n-1}\sigma_n$$
$$(a_{n+1}\sigma_{j+1}..\sigma_2\sigma_1\sigma_{j+2}..\sigma_{n-1}\sigma_n)^{m-(p+1)}$$
$$a_{n+1}\sigma_{j+1}..\sigma_{d_1}\sigma_{j+2}..\sigma_{d_2}\sigma_{j+3}..\sigma_{d_3} \ \dots\ \sigma_{j+z}..\sigma_{d_z}.$$

Where $0 \leq i_1 < i_2 \ \dots\ < i_p < r_p < r_2 \ \dots\ < r_1 \leq n+1$ and $r_p - i_p \geq 4$.

With $i_p < j$ and $j + 2 < r_p$.

While $d_1 < d_2 \ \dots\ < d_z$ and $i_1 + c \geq d_c$ for $1 \leq c \leq z$.

$$c_4 = \sigma_{i_1}..\sigma_2\sigma_1\sigma_{r_1}..\sigma_{n-1}\sigma_n a_{n+1}\sigma_{i_2}..\sigma_2\sigma_1\sigma_{r_2}..\sigma_{n-1}\sigma_n a_{n+1} \ \dots\ \sigma_{i_p}..\sigma_2\sigma_1\sigma_{r_p}$$
$$a_{n+1}\sigma_{l_1}..\sigma_{g_1}\sigma_{l_2}..\sigma_{g_2} \ \dots\ \sigma_{l_t}..\sigma_{g_t}$$

Where: $0 \leq i_1 < i_2 \ldots < i_p < r_p < r_2 \ldots < r_1 \leq n$ *and* $r_p - i_p \geq 3$.

With $1 \leq l_1 < l_2 .. < l_t \leq n$ *and* $1 \leq g_1 < g_2 .. < g_t \leq n$.

While $i_p < l_1$, $g_t < r_p$ *and* $g_k \leq l_k$ *for any* $1 \leq k \leq n$.

In all cases p is necessarily bounded by $n/2$ *.*

In order to get to the final form of our classification we shall do one more step, explained in the following remark:

Remarks 3.4.16. *We set* $\sigma_t = 1$ *when* $0 \geq t$ *or* $t \geq n+1$. *We can actually unify cases* c_1, c_2 *and* c_3 *with the following formula:*

$$\sigma_{i_1}..\sigma_2\sigma_1\sigma_{r_1}..\sigma_{n-1}\sigma_n a_{n+1}\sigma_{i_2}..\sigma_2\sigma_1\sigma_{r_2}..\sigma_{n-1}\sigma_n a_{n+1} \ldots \sigma_{i_p}..\sigma_2\sigma_1\sigma_{r_p} \ldots \sigma_{n-1}\sigma_n$$
$$(a_{n+1}\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}\sigma_n)^K$$
$$a_{n+1}\sigma_j..\sigma_{d_1}\sigma_{j+1}..\sigma_{d_2}\sigma_{j+2}..\sigma_{d_3} \ldots \sigma_{j+z}..\sigma_{d_{z+1}}.$$

Where $0 \leq i_1 < i_2 \ldots < i_p < r_p < r_2 \ldots < r_1 \leq n+1$ *and* $r_p - i_p \geq 2$.

With $i_p < j$, $j \leq r_p - 1$, $i_1 \leq n$ *and* $1 \leq K$.

While $d_1 < d_2 \ldots < d_{z+1}$ *and* $j + c \geq d_{c+1}$ *for* $0 \leq c \leq z$.

Unifying c_4 (we mean the case $k = 0$) with the the last cases would give our general formula a better shape, unfortunately the residue element can have more possibilities!

Moreover, we see that our formula expresses the elements of $W^c(\tilde{A}_2)$. With this last remark, the proof of theorem 3.4.4 is officially done, after noticing that the way in which we get the general form, ensures the uniqueness of this form.

3.5 More about the structure of $B(\tilde{A}_n)$

Now we consider the tower of affine braid groups:

$$B(\tilde{A}_0) \xrightarrow{F_0} B(\tilde{A}_1) \xrightarrow{F_1} \ldots B(\tilde{A}_{n-1}) \xrightarrow{F_n} B(\tilde{A}_n) \xrightarrow{F_{n+1}} \ldots$$

Where $B(\tilde{A}_0)$ is the trivial group. Via F_n, every $B(\tilde{A}_{n-1})$ injects into $B(\tilde{A}_n)$ for $0 \leq n$. We are interested with viewing $B(\tilde{A}_n)$ containing $B(\tilde{A}_{n-1})$. The following computations are done in view of understanding the tower of affine Temperley-Lieb algebras, which will be treated in section 5.

In what follows we show that a given fully commutative element in $W(\tilde{A_n})$ has a kind of "canonical" form. We suppose that the affine length of this element is strictly greater than 1. Since we use only braid relations, we can lift our element to the element in $B(\tilde{A_n})$ which has the same expression (keeping the same symbols for the generators of the affine braid group, and their images via the natural surjection onto affine Coxeter group).

Let $1 \leq n$. Let $\bar{w}$ be in $W^c(\tilde{A_n})$. The general form of $\bar{w}$ is

$$\bar{w} = \sigma_{i_1}..\sigma_2\sigma_1\sigma_{r_1}..\sigma_{n-1}\sigma_n a_{n+1}\sigma_{i_2}..\sigma_2\sigma_1\sigma_{r_2}..\sigma_{n-1}\sigma_n a_{n+1} \ ... \ \sigma_{i_p}..\sigma_2\sigma_1\sigma_{r_p} \ ... \ \sigma_{n-1}\sigma_n$$
$$(a_{n+1}\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}\sigma_n)^k\bar{u},$$

where $0 \leq i_1 < i_2 \ ... \ < i_p < r_p < r_2 \ ... \ < r_1 \leq n+1$ and $r_p - i_p \geq 2$,

with $i_p < j$, $j \leq r_p - 1$, $i_1 \leq n$ and $0 \leq k$,

while $\bar{u} = a_{n+1}\bar{v}$, with v fully commutative in $W(A_n)$.

We lift $\bar{w}$,(resp. $\bar{u}$ and $\bar{v}$) to w,(resp. u and v) in $B(\tilde{A_n})$, by corollary 3.1.5. Assume that $j < n$, i.e., w is not of the form $v(a_{n+1}\sigma_n..\sigma_2\sigma_1)^k u$. We show that w has the form:

$h(\sigma_n..\sigma_1 a_{n+1})^m x$, where x is in $B(A_n)$ and h is in $B(\tilde{A_{n-1}})$.

We show as well that $\bar{w}$ has the form:

$\bar{f}(\sigma_n..\sigma_1 a_{n+1})^m \sigma_n..\sigma_i$, where $1 \leq i \leq n+1$, m is a positive integer and $\bar{f}$ is in $W(\tilde{A_{n-1}})$.

Recall that $\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}$ acts on the elements of $B(\tilde{A_{n-1}})$ exactly the way ϕ_n^{-1} does in $B(B_n)$. In order to simplify, we set $\phi_n^{-1} = \psi$. We write $(\sigma_n..\sigma_1 a_{n+1})^d h = \psi^d[h](\sigma_n..\sigma_1 a_{n+1})^d$, for any h in $B(\tilde{A_{n-1}})$. The automorphism ϕ is of order n, of course, d is to be taken mod$n$. We keep in mind that $a_{n+1}a_n = a_n\sigma_n = \sigma_n a_{n+1}$.

Lemma 3.5.1. *Let y be $\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}\sigma_n a_{n+1}$ in $B(\tilde{A_n})$. Let $w = y^k$ with $2 \leq j \leq n-1$ and $2 \leq k$. Suppose that $k = m(n-j+1) + r$ where $0 \leq r < n-j+1$. Then:*

(1) *If $m = 0$ we have:*

$$w = (\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}a_n)^r\sigma_n\sigma_{n-1}..\sigma_{n+1-r}.$$

(2) *if* $0 < m$ *we have:*

$$w = \prod_{i=0}^{i=m-1} \psi^i \left[(\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}a_n)^{n-j}\sigma_j..\sigma_{n-1} \right] \psi^m \left[(\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}a_n)^r \right]$$
$$(\sigma_n\sigma_{n-1}..\sigma_2\sigma_1a_{n+1})^m\sigma_n\sigma_{n-1}..\sigma_{n+1-r}.$$

Proof.

$$\begin{aligned} \text{We have } y &= \sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}\sigma_na_{n+1} = \sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}a_n\sigma_n. \\ \text{Hence, } y^2 &= \sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}a_n\sigma_n\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}\sigma_na_{n+1} \\ &= \sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}a_n\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}\sigma_na_{n+1}\sigma_{n-1} \\ &= (\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}a_n)^2\sigma_n\sigma_{n-1}. \end{aligned}$$

Continuing this way, we can see that whenever $0 \leq r \leq n-j$:

$$y^r = (\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}a_n)^r\sigma_n\sigma_{n-1}..\sigma_s, \text{ with } s+r = n+1. \text{ in particular:}$$

$$y^{n-j} = (\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}a_n)^{n-j}\sigma_n\sigma_{n-1}..\sigma_{j+1}. \text{ Thus:}$$

$$\begin{aligned} y^{n-j+1} &= (\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}a_n)^{n-j}\sigma_n\sigma_{n-1}..\sigma_{j+1}\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}\sigma_na_{n+1} \\ &= (\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}a_n)^{n-j}\sigma_n\sigma_{n-1}..\sigma_{j+1}\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}\sigma_na_{n+1} \\ &= (\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}a_n)^{n-j}\sigma_j..\sigma_{n-1}\sigma_n\sigma_{n-1}..\sigma_2\sigma_1a_{n+1}. \text{ We see that:} \end{aligned}$$

$$\begin{aligned} y^{2(n-j+1)} &= \psi^0 \left[(\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}a_n)^{n-j}\sigma_j..\sigma_{n-1} \right] \\ &\quad .\psi^1 \left[(\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}F(a_n))^{n-j}\sigma_j..\sigma_{n-1} \right] (\sigma_n\sigma_{n-1}..\sigma_2\sigma_1a_{n+1})^2. \end{aligned}$$

In the same way, for $m \geq 0$, considering the action of $\sigma_n\sigma_{n-1}..\sigma_2\sigma_1a_{n+1}$ on $B(\tilde{A}_{n-1})$, we see that:

$$y^{m(n-j+1)} = \prod_{i=0}^{i=m-1} \psi^i \left[(\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}F(a_n))^{n-j}\sigma_j..\sigma_{n-1} \right] (\sigma_n\sigma_{n-1}..\sigma_2\sigma_1a_{n+1})^m.$$

Finally, let $k = m(n-j+1)+r$, where $0 \leq r < n-j+1$. We have:

$$y^k = \prod_{i=0}^{i=m-1} \psi^i \left[(\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}F(a_n))^{n-j}\sigma_j..\sigma_{n-1}\right]$$

$$(\sigma_n\sigma_{n-1}..\sigma_2\sigma_1 a_{n+1})^m(\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}a_n)^r\sigma_n\sigma_{n-1}..\sigma_{n+1-r}.$$ Thus,

$$y^k = \prod_{i=0}^{i=m-1} \psi^i \left[(\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}F(a_n))^{n-j}\sigma_j..\sigma_{n-1}\right] \psi^m \left[(\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}a_n)^r\right]$$

$$(\sigma_n\sigma_{n-1}..\sigma_2\sigma_1 a_{n+1})^m\sigma_n\sigma_{n-1}..\sigma_{n+1-r}.$$

□

In particular, for $j = 1$, i.e., $w = (\sigma_1\sigma_2..\sigma_n a_{n+1})^k$, we have:

$$y^k = \prod_{i=0}^{i=m-1} \psi^i \left[(\sigma_1..\sigma_{n-1}a_n)^{n-1}\sigma_1..\sigma_{n-1}\right] \psi^m \left[(\sigma_1..\sigma_{n-1}a_n)^r\right] (\sigma_n..\sigma_1 a_{n+1})^m\sigma_n\sigma_{n-1}..\sigma_i.$$

Now we go back to the general form of $\bar{w}$, that is:

$$\bar{w} = \sigma_{i_1}..\sigma_2\sigma_1\sigma_{r_1}..\sigma_{n-1}\sigma_n a_{n+1}\sigma_{i_2}..\sigma_2\sigma_1\sigma_{r_2}..\sigma_{n-1}\sigma_n a_{n+1} \ ... \ \sigma_{i_p}..\sigma_2\sigma_1\sigma_{r_p} \ ... \ \sigma_{n-1}\sigma_n$$
$$(a_{n+1}\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}\sigma_n)^k\bar{u},$$

where $0 \leq i_1 < i_2 \ ... \ < i_p < r_p < r_2 \ ... \ < r_1 \leq n+1$ and $r_p - i_p \geq 2$,

with $i_p < j$, $j \leq r_p - 1$, $i_1 \leq n$ and $0 \leq k$,

while $\bar{u} = a_{n+1}\bar{v}$, with v fully commutative in $W(A_n)$.

Then, $$w = \sigma_{i_1}..\sigma_2\sigma_1\sigma_{r_1}..\sigma_{n-1}\sigma_n a_{n+1}\sigma_{i_2}..\sigma_2\sigma_1\sigma_{r_2}..\sigma_{n-1}\sigma_n a_{n+1} \ ... \ \sigma_{i_p}..\sigma_2\sigma_1\sigma_{r_p} \ ... \ \sigma_{n-1}\sigma_n a_{n+1}$$
$$(\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}\sigma_n a_{n+1})^k v,$$

with conditions similar to those above. In particular, v is fully commutative in $W(A_n)$.

We see that $r_i \leq r_1 - i$, but $r_1 \leq n+1$. which gives: $r_i \leq n-i+1$ for all i.

Here we treat two main cases:

- $r_1 \leq n$, that is σ_n belongs to the support of $\sigma_{i_1}..\sigma_2\sigma_1\sigma_{r_1}..\sigma_{n-1}\sigma_n$.

- $r_1 = n+1$, that is σ_n belongs to the support of $\sigma_{i_1}..\sigma_2\sigma_1\sigma_{r_1}..\sigma_{n-1}\sigma_n$, (this case covers the case where $p=0$).

We start by the first case $r_1 \leq n$ (we suppose that $p > 0$).

Set $x := \sigma_{i_1}..\sigma_2\sigma_1\sigma_{r_1}..\sigma_{n-1}\sigma_n a_{n+1}\sigma_{i_2}..\sigma_2\sigma_1\sigma_{r_2}..\sigma_{n-1}\sigma_n a_{n+1}\ ...\ \sigma_{i_p}..\sigma_2\sigma_1\sigma_{r_p}\ ...\ \sigma_{n-1}\sigma_n a_{n+1}$.

Thus, $w = x(\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}\sigma_n a_{n+1})^k v$. Repeating the first step of lemma 2.5.1 (keeping in mind that $r_i \leq n-i$), we see that:

$$\begin{aligned} x &= \sigma_{i_1}..\sigma_2\sigma_1\sigma_{r_1}..\sigma_{n-1}a_n\sigma_{i_2}..\sigma_2\sigma_1\sigma_{r_2}..\sigma_{n-1}\sigma_n a_{n+1}\ ...\ \sigma_{i_p}..\sigma_2\sigma_1\sigma_{r_p}\ ...\ \sigma_{n-1}\sigma_n a_{n+1}\sigma_{n-(p-1)} \\ &= \sigma_{i_1}..\sigma_2\sigma_1\sigma_{r_1}..\sigma_{n-1}a_n\sigma_{i_2}..\sigma_2\sigma_1\sigma_{r_2}..\sigma_{n-1}a_n\ ...\ \sigma_{i_p}..\sigma_2\sigma_1\sigma_{r_p}\ ...\ \sigma_{n-1}\sigma_n a_{n+1}\sigma_{n-p}\sigma_{n-(p-1)}. \end{aligned}$$

After p steps we see that:

$$x = \sigma_{i_1}..\sigma_2\sigma_1\sigma_{r_1}..\sigma_{n-1}a_n\sigma_{i_2}..\sigma_2\sigma_1\sigma_{r_2}..\sigma_{n-1}a_n..\sigma_{i_p}..\sigma_2\sigma_1\sigma_{r_p}\ ...\ \sigma_{n-1}a_n\sigma_n\sigma_{n-1}..\sigma_{n-(p-1)}.$$

Now, we set $\epsilon := n-(p-1)$. We show that $\epsilon > j+1$ as follows.

We know that $j+1 \leq r_p$, but $r_p < n-p < n-(p-1)$. In other words:

$$j+1 \leq r_p < n-(p-1), \text{ that is } j+1 < \epsilon.$$

Set $\rho := \sigma_{i_1}..\sigma_2\sigma_1\sigma_{r_1}..\sigma_{n-1}a_n\sigma_{i_2}..\sigma_2\sigma_1\sigma_{r_2}..\sigma_{n-1}a_n\ ...\ \sigma_{i_p}..\sigma_2\sigma_1\sigma_{r_p}\ ...\ \sigma_{n-1}a_n$. Thus,

$$w = \rho\sigma_n\sigma_{n-1}..\sigma_\epsilon(\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}\sigma_n a_{n+1})^k v, \text{ with } \rho \in B(\tilde{A_{n-1}}) \text{ and } \epsilon > j+1.$$

Now we have $w = \rho\sigma_n\sigma_{n-1}..\sigma_\epsilon y^k v$. Every y acts on σ_i in the following way: $\sigma_i y = y\sigma_{i-1}$, for $\epsilon \leq i \leq n$, since $j+1 < \epsilon$ and hence $j+1 < i$.

If $k=0$, the job is done (this case is included in the general form).

Let $1 \leq k$. We have two main cases:

(1) $1 \leq k < \epsilon-(j+1)$.

(2) $\epsilon-(j+1) \leq k$.

We start by (1). Set $e := \epsilon - k$. We have:

$$\begin{aligned} &\sigma_n\sigma_{n-1}..\sigma_\epsilon y^k = y^k\sigma_{e+(n-\epsilon)}\sigma_{e+(n-\epsilon)-1}..\sigma_e. \text{ That is,} \\ &\sigma_n\sigma_{n-1}..\sigma_\epsilon y^k = y^k\sigma_{n-k}\sigma_{n-1-k}..\sigma_{\epsilon-k}. \end{aligned}$$

We have $k < \epsilon - (j+1) = n - (p-1) - (j+1) = n - j - p < n - j$. Thus, in the terms of lemma 3.5.1, we are in case (1), (even with the same y). That is:

$$y^k = (\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}a_n)^k\sigma_n\sigma_{n-1}..\sigma_{n+1-k}.$$

$$\text{Thus } w = \rho(\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}a_n)^k\sigma_n\sigma_{n-1}..\sigma_{n-1-k}\sigma_{n-k}\sigma_{n-1-k}..\sigma_{\epsilon-k}v$$

which is basically the case $m = 0$.

Case (2), where $\epsilon - (j+1) \leq k$.

We see that for $k = \epsilon - (j+1)$, we have:

$$\sigma_n\sigma_{n-1}..\sigma_\epsilon y^k = y^k\sigma_{n-k}\sigma_{(n-k)-1}..\sigma_{j+1}. \text{ In other terms:}$$

$$\sigma_n\sigma_{n-1}..\sigma_\epsilon(y)^{\epsilon-(j+1)} = (y)^{\epsilon-(j+1)}\sigma_{j+p}\sigma_{j+p-1}..\sigma_{j+1}.$$

$$\text{That is } \sigma_n\sigma_{n-1}..\sigma_\epsilon y^k = \sigma_n\sigma_{n-1}..\sigma_\epsilon y^{\epsilon-(j+1)}y^{k-(\epsilon-(j+1))}.$$

Now, $\epsilon - (j+1) = n - (p-1) - (j+1) = n - j - p < n - j$. Here, we can apply lemma 3.5.1 (again we are in the first case). Precisely :

$$\begin{aligned}(y)^{\epsilon-(j+1)} &= (\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}a_n)^{\epsilon-(j+1)}\sigma_n\sigma_{n-1}..\sigma_{n+1-(\epsilon-(j+1))}\\ &= (\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}a_n)^{\epsilon-(j+1)}\sigma_n\sigma_{n-1}..\sigma_{j+p+1}.\end{aligned}$$

Set $h := k - (\epsilon - (j+1))$. We get:

$$\sigma_n\sigma_{n-1}..\sigma_\epsilon y^k = (\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}a_n)^{\epsilon-(j+1)}\sigma_n\sigma_{n-1}..\sigma_{j+p+1}\sigma_{j+p}..\sigma_{j+1}yy^{h-1}.$$

$$\text{But, } \sigma_n\sigma_{n-1}..\sigma_{j+p+1}\sigma_{j+p}..\sigma_{j+1}yy^{h-1} = \sigma_n..\sigma_{j+1}\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_n a_{n+1}y^{h-1},$$

which is equal to $\sigma_n\sigma_{n-1}..\sigma_1\underbrace{\sigma_{j+1}..\sigma_{n-1}\sigma_n}a_{n+1}y^{h-1}$,

Which is $\underbrace{\sigma_j..\sigma_{n-2}\sigma_{n-1}}\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}y^{h-1}$.

Now set $\eta := \rho\sigma_j..\sigma_{n-2}\sigma_{n-1} \in B(\tilde{A}_{n-1})$. We get $w = \eta\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}y^{h-1}v$.

(a) If $h-1 \leq n-j$, we see that:

$$\begin{aligned} w &= \eta\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}(\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}a_n)^{h-1}\sigma_n\sigma_{n-1}..\sigma_{n+1-(h-1)}v \\ &= \eta\psi\left[(\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}a_n)^{h-1}\right]\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}\sigma_n\sigma_{n-1}..\sigma_{n+1-(h-1)}v. \end{aligned}$$

(b) If $n-j < h-1$, we see that:

$$\begin{aligned} w = \eta\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1} &\prod_{i=0}^{i=m-1}\psi^i\left[(\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}a_n)^{n-j}\sigma_j..\sigma_{n-1}\right] \\ &\psi^m\left[(\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}a_n)^r\right](\sigma_n\sigma_{n-1}..\sigma_2\sigma_1 a_{n+1})^m\sigma_n\sigma_{n-1}..\sigma_{n+1-r}v. \end{aligned}$$

Thus, $w = \eta\prod_{m=1}^{i=m}\psi^i\left[(\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}a_n)^{n-j}\sigma_j..\sigma_{n-1}\right]\psi^{m+1}\left[(\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}a_n)^r\right]v$
$(\sigma_n\sigma_{n-1}..\sigma_2\sigma_1 a_{n+1})^{m+1}\sigma_n\sigma_{n-1}..\sigma_{n+1-r}v,$

where $h-1 = m(n-j+1)+r$, with $0 \leq r < n-j+1$.

So in this case (namely, $r_1 \leq n$), we see that w is written as

$$c(\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1})^k\sigma_n\sigma_{n-1}..\sigma_i,$$

where c is in $B(\tilde{A_{n-1}})$, $1 \leq i \leq n+1$ and $0 \leq k$.

Now, we deal with the second main case: $r_1 = n+1$. Here σ_n is not in the support of $\sigma_{i_1}..\sigma_2\sigma_1\sigma_{r_1}..\sigma_{n-1}\sigma_n$ (which is equal in this case to $\sigma_{i_1}..\sigma_2\sigma_1$). Hence, we can suppose that the element in question is of the form $\sigma_{i_1}..\sigma_2\sigma_1 a_{n+1}w$, where $0 \leq r_0 < r_1$. Here we have $r_0 < n$, since $r_0 = n$ is the case of positive powers of $\sigma_n..\sigma_1\sigma_1 a_{n+1}$. Moreover, when $r_0 = 0$, then the element in question is of the form $a_{n+1}w$, which is the case $p = 0$. As a consequence of this discussion we get the following corollary.

Corollary 3.5.2. *Let $\bar{w}$ be fully commutative in $W(\tilde{A}_n)$, where $2 \leq n$. Let w be the corresponding element in $B(\tilde{A}_n)$, as above. Then w can be written in one and only one of the following two forms:*

$$\begin{gathered} c(\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1})^k v, \\ \text{or } \sigma_{i_0}..\sigma_2\sigma_1 a_{n+1}c(\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1})^k v. \end{gathered}$$

Here, c is in $B(\tilde{A_{n-1}})$, while v is in $B(A_n)$.

Moreover, $\bar{w}$ can be written in one, and only one of the following two forms (deduced from the above two forms, considering the left classes of $W(A_{n-1})$ in $W(A_n)$):

$$\bar{d}(\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1})^k\sigma_n\sigma_{n-1}..\sigma_i,$$
$$\text{or } \sigma_{i_0}..\sigma_2\sigma_1 a_{n+1}\bar{d}(\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1})^k\sigma_n\sigma_{n-1}..\sigma_i.$$

Here $\bar{d}$ is in $W(\tilde{A_{n-1}})$, where $1 \le i \le n$, with $0 \le r_0 \le n-1$ and $0 \le k$.

3.6 Bibliographical remarks and problems

We followed [Bou81] and [Lus99] in the definitions and results of 3.1.

In fact, the problem of passing from a semi-direct product presentation of a given element to a reduced expression still stands! An algorithm solving the problem would be of a great use, on the way of a better understanding of the group $W(\tilde{A}_n)$.

In theorem 3.4.2, we give a normal form for fully commutative elements, which is basically, depending on viewing any of such elements as a product of blocks of non-commuting generators, the expression is reduced, hence, for any given element w, we can deduce, from its normal form, the set $\mathscr{L}(w)$ directly. We recall in [FG99], the left decomposition:

> We set $P = \{U \subset \Gamma\} \cup \{\emptyset\}$, where U consists of non-adjacent vertices. Let $i(U)$ be the product of the elements of U. It is well defined, for these elements commute to each other other. If w is any element in $W^c(\tilde{A}_n)$, then w can be uniquely written as $w = i(G_1)...i(G_m)$ being a reduced expression. Where $G_k = \mathscr{L}(i(G_{k-1})...i(G_1)w) \in P$, with $G - m \neq \emptyset$.

We see that this decomposition presents w as a product of blocks of commuting generators. Nevertheless, it does not give a explicit expression of w. In this paper, the authors made use of the theory of cells to show the faithfulness of the diagrammatic presentation of affine T-L algebra. Many of these results, such as the property R, could be shown using our normal form. An interesting attempt, would be to pass from the decomposition of elements (such as M_1 and M_2) to the normal form given in this work. We are confident that many interesting results would follow from a bridge between this paper and our normal form.

In remark 3.4.14, we distinguished between w_1 and w_2, actually in our work we did not make any difference, but, some times we distinguish between the two cases: even and odd n, we have as example [FG99], in particular, when classifying the two-sided cells.

بمهندٍ ماضٍ ورمحٍ أسمر. عنترَ عبس

With a decent blade, a brown ..morphism

Chapter 4

Iwahori-Hecke Algebras and Markov traces

Let K be a integral domain of characteristic 0. In what follows q_s, Q, q, q' and q'' are invertible elements. In some subsection K is to be considered a field, it will be announced once needed, or direct from the contest.

Let (W, S) be a Coxeter system. The Hecke algebra is a deformation of the group algebra $K[W]$, coming from the theory of representations of finite groups of Lie type, and many other branches of mathematics. The Hecke algebras has many definitions. We will give a definition coming from Coxeter systems, then we will view the Hecke algebra as a quotient of the braid groups algebra.

Finite dimensional Hecke algebras, i.e., Hecke algebras related to spherical types of Coxeter systems, are to be defined first, we mention some facts and results about these algebras and their Markov traces without proofs. We will then, focus on the $\tilde{A}$-type Hecke algebras, and put them of a tower of algebras.

After fixing (W, S), we consider K and q_s for $s \in S$ as above, such that $q_r = q_t$ whenever t and r are conjugate in W.

Definition 4.0.1. *We define the Iwahori-Hecke algebra of (W, S) over K with parameters q_s (denoted by $H(W, S, (q_s))$, or simply $H(W)$) as the algebra with unit, over K, generated by the set $\{g_s : s \in S\}$, with the following defining relations:*

- $\prod(g_s, g_t : m_{st}) = \prod(g_t, g_s : m_{st})$ *for any non-equal s, t in S with $m_{s,t} \neq \infty$.*
- $g_s^2 = (q_s - 1)g_s + q_s 1$ *for any s in S.*

Let $g_w := g_{s_1} g_{s_2} .. g_{s_m}$, where $w = s_1 s_2 .. s_m$ be a reduced expression in (W, S). Then by theorem 3.1.4 g_w is well defined, i.e., g_w is independent of the choice of the reduced expression of w. Notice that specializing the parameters q_s to 1, gives the group algebra $K[W]$.

Theorem 4.0.2. *[Bou81]. Let $H(W)$ be the Hecke algebra associated to a Coxeter system (W,S). Let $I \subseteq S$. Then:*

- *$\{g_w; w \in W\}$ is a K-basis of $H(W)$.*
- *For u, v in W and s in S we have:*

$$g_u g_w = g_{uv} \text{ when } l(uv) = l(u) + l(v),$$
$$g_u g_s = (q_s - 1)e_u + q_s g_{us} \text{ when } l(us) = l(u) - 1.$$

- *The K-sub-vector space of $H(W)$, spanned by $(g_w)_{w \in W_I}$, is a subalgebra of $H(W)$. this subalgebra is isomorphic to $H(W_I)$. This isomorphism is compatible with embedding of W_I in W. We call such subalgebra a parabolic subalgebra of $H(W)$.*

Remark 4.0.3. *In what follows, suppose that A is an algebra, suppose that x is in A. We mean by the ideal generated by x: the two-sided ideal generated by x, we denote it $\langle x \rangle_A$, or $\langle x \rangle$, if there is no ambiguity. While we denote $\langle x \rangle_A^{alg}$ the subalgebra generated by x.*

4.1 The A-type Hecke algebra

Consider the group algebra $K[B(A_n)]$. Here, we take $K = \mathbb{Q}[q^{\pm 1}]$. We consider the ideal I_n, generated by the elements $Z_i := \sigma_i^2 - (q-1)\sigma_i - q$, for $1 \leq i \leq n$. Notice that for some $1 \leq j \leq n-1$ we have:

$$\sigma_j \sigma_{j+1} Z_j \sigma_{j+1}^{-1} \sigma_j^{-1} = \sigma_j \sigma_{j+1} (\sigma_j^2 - (q-1)\sigma_j - q) \sigma_{j+1}^{-1} \sigma_j^{-1}, \text{ this is equal to}$$

$$\sigma_j \sigma_{j+1} \sigma_j^2 \sigma_{j+1}^{-1} \sigma_j^{-1} - (q-1)\sigma_j \sigma_{j+1} \sigma_j \sigma_{j+1}^{-1} \sigma_j^{-1} - q, \text{ which is equal to}$$

$$\sigma_{j+1}^2 - (q-1)\sigma_{j+1} - q.$$

In other terms, the Z_i are conjugate, thus $I_n = \langle \sigma_1^2 - (q-1)\sigma_1 - q \rangle_{K[B(A_n)]}$.

We can view $H_n(q)$ as the quotient $K[B(A_n)]/I_n$. Via the natural surjection, we set g_{σ_i} to be the image of σ_i for $1 \leq i \leq n$. $\{g_w; w \in W(A_n)\}$ forms a K-basis of $H_n(q)$. Moreover, theorem 3.0.2 ensures that the tower of groups (assuming that $W(A_0) = 1$):

$W(A_0) \subset W(A_1).. \subset W(A_{n-1}) \subset W(A_n)..$ induces the following tower of algebras
$H_0(q) \ \subset H_1(q) \ .. \subset H_{n-1}(q) \ \subset H_n(q)..$

Obviously $H_0(q) = K$. On the other hand the injection x_n of $B(A_{n-1})$ in $B(A_n)$ becomes by linearity, an injection of $K[B(A_{n-1})]$ in $K[B(A_n)]$, which obviously, gives the following injection (we keep the same notation x_n):

$$\begin{aligned} x_n : H_{n-1}(q) &\longrightarrow H_n(q) \\ g_{\sigma_i} &\longmapsto g_{\sigma_i} \text{ for } 1 \leq i \leq n-1. \end{aligned}$$

Moreover, $x_n(g_w) = g_w$ for any w in $W(A_{n-1})$. The multiplication is given as follows (for any u and v in $W(A_n)$, and $1 \leq i \leq n$):

$$\begin{aligned} g_u g_w &= g_{uv} && \text{when } l(uv) = l(u) + l(v), \\ g_u g_{\sigma_i} &= (q-1)g_u + q g_{u\sigma_i} && \text{when } l(u\sigma_i) = l(u) - 1. \end{aligned}$$

Any element of this basis is invertible, since the generators are; $g_{\sigma_i}^{-1} = \frac{1}{q} g_{\sigma_i} + \frac{1-q}{q}$.

4.1.1 Basis

Let $H_n(q)$ be as above. In this subsection, we take $K = \mathbb{Q}[q'^{\pm 1}, q''^{\pm 1}]$. Now, we have the K-basis $\{g_w; w \in W(A_n)\}$. In a given representation of $H_n(q)$, each generator has q and 1 as eigenvalues. Now we change the generators, so that each one has two eigenvalues. Suppose that q is written in K as $q = -\frac{q'}{q''}$, then, we set $T_{\sigma_i} := -q'' g_{\sigma_i}$ for $1 \leq i \leq n$. Hence, $T_w = (-q'')^{l(w)} g_w$. We see directly that the set $\{T_w; w \in W(A_n)\}$ is a K-basis of $H_n(q)$. We set $S := q' + q''$ and $P := q'q''$, we find the multiplication law of the new basis takes the form:

$$\begin{aligned} T_w T_v &= T_{wv} && \text{when } l(wv) = l(w) + l(v). \\ T_{\sigma_i} T_w &= ST_w - PT_{\sigma_i w} && \text{when } l(w\sigma_i) = l(w) - 1. \end{aligned}$$

for any for w, v in $W(A_n)$ and s in $\{\sigma_1, .., \sigma_n\}$ .More, $T_{\sigma_i}^{-1} = \frac{1}{P}(S - T_{\sigma_i})$.

Call π'_n the surjection above, that is $\pi'_n : K[B(A_n)] \longrightarrow H_n(q)$. It sends σ_i to g_{σ_i}, hence to $-\frac{1}{q''} T_{\sigma_i}$. We consider π_n the surjective from $K[B(A_n)]$ onto $H_n(q)$, which sends σ_i to T_{σ_i}.

4.1.2 Markov trace

The following lemma is used in the construction of Markov trace (see [Jon87]):

Lemma 4.1.1. *The linear application*

$$\begin{aligned} H_n(q) \oplus H_n(q) \otimes_{H_{n-1}(q)} H_n(q) &\longrightarrow H_{n+1}(q) \\ a + b \otimes c &\longmapsto a + bT_{\sigma_n} c, \end{aligned}$$

is an $H_n(q)$-module-$H_n(q)$ isomorphism.

The following theorem is due to Jones. We give it slightly different from which appears in literature, for example [GP05]:

Theorem 4.1.2. *Under the above notations, we have:*

- *There exists a unique collection of traces $\tau_1, \tau_2..\tau_n...$, such that:*

  (1) $\tau_{n+1} : H_n(q) \longrightarrow K$ *is a trace* $\forall 0 \leq n$.

  (2) $\tau_1(1) = 1$.

  (3) $\tau_{n+1}(hT^{\pm 1}_{\sigma_n}) = \tau_n(h)$, *for any h in $H_{n-1}(q)$.*

- *For $1 \leq n$, every $\tau_{n+1} : H_n(q) \longrightarrow K$ is given by*
$$\tau_{n+1}(a + bT_{\sigma_n}c) = \frac{P+1}{S}\tau_n(a) + \tau_n(bc).$$
- *The values of every τ_{n+1} are Laurent polynomials in S and P.*

Definition 4.1.3. *The above collection $(\tau_i)_{1\leq i}$, is called a Markov trace over the tower of Hecke algebras. The element τ_n of the collection, is to be called the n-th Markov trace.*

Remark 4.1.4. *For h in $H_{n-1}(q)$, we see that:*
$$\tau_{n+1}\Big(hg_{\sigma_n}\Big) = \frac{-1}{q''}\tau_{n+1}\Big(hT_{\sigma_n}\Big) = \frac{-1}{q''}\tau_n\Big(h\Big).$$

While,
$$\begin{aligned}
\tau_{n+1}\Big(hg^{-1}_{\sigma_n}\Big) &= \frac{1}{q}\tau_{n+1}\Big(hg_{\sigma_n}\Big) + \frac{1-q}{q}\tau_{n+1}\Big(h\Big) \\
&= \frac{-1}{qq''}\tau_{n+1}\Big(hT_{\sigma_n}\Big) + \Big(\frac{1-q}{q}\Big)\Big(\frac{P+1}{S}\Big)\tau_n(h) \\
&= \Bigg[\frac{-1}{qq''} + \Big(\frac{1-q}{q}\Big)\Big(\frac{P+1}{S}\Big)\Bigg]\tau_n(h) \\
&= \Bigg(\frac{-(q'q''+1)}{q'}\Bigg)\tau_n(h).
\end{aligned}$$

Hence,
$$\begin{aligned}
\tau_{n+1}\Big(hg_{\sigma_n}\Big) &= \frac{-1}{q''}\tau_n\Big(h\Big), \text{ and} \\
\tau_{n+1}\Big(hg^{-1}_{\sigma_n}\Big) &= -q''\tau_n\Big(h\Big).
\end{aligned}$$

In fact we could define Markov trace as above, but with replacing the condition (3) : $\tau_{n+1}\Big(hT^{\pm 1}_{\sigma_n}\Big) = \tau_n\Big(h\Big)$, *for any* h *in* $H_{n-1}(q)$, *by the condition (3')* : $\tau_{n+1}\Big(hg_{\sigma_n}\Big) = \frac{-1}{q''}\tau_n\Big(h\Big)$ *and* $\tau_{n+1}\Big(hg^{-1}_{\sigma_n}\Big) = -q''\tau_n\Big(h\Big)$*! Often in the literature* $(\tau_n)_{1\leq n}$ *is defined with the condition* $\tau_{n+1}\Big(hg^{\pm 1}_{\sigma_n}\Big) = \tau_n\Big(h\Big)$. *This should not confuse the reader, since, in this, case* q'' *is implicitly equal to -1! hence, (3) and (3') become the same statement.*

We should here notice that in the case where q'' *is specialized to -1, some authors prefer to write this condition as* $\tau_{n+1}\Big(hg_{\sigma_n}\Big) = \tau_n\Big(h\Big)$, *i.e., prefer to omit the condition* $\tau_{n+1}\Big(hg^{-1}_{\sigma_n}\Big) = \tau_n\Big(h\Big)$. *One should not think of this condition as a result of* $\tau_{n+1}\Big(hg_{\sigma_n}\Big) = \tau_n\Big(h\Big)$, *implicitly it exists and it is independent, so that the trace would be unique.*

Set $H(q) := \bigcup_{0\leq i} H_i(q)$. Now define and $\tau : H(q) \longrightarrow K$ by:

$$\tau\Big(x\Big) = \Big(\frac{P+1}{S}\Big)^{-n}\tau_{n+1}(x), \text{ for any } x \in H_n(q) \text{ with } 0 \leq n.$$

We can see directly that:

(1) τ is a trace function.

(2) $\tau(1) = 1$.

(3) $\tau(hT_{\sigma_n}) = (\frac{P+1}{S})^{-1}\tau(h)$ for $n \geq 1$ and $h \in H_n(q)$.

And vice-versa, suppose that we have some z in K, there exists $\tau : H(q) \longrightarrow K$ such that

(1) τ is a trace function.

(1) $\tau(1) = 1$.

(1) $\tau(hT_{\sigma_n}) = z\tau(h)$ for $n \geq 1$ and $h \in H_n(q)$.

Now this trace is uniquely determined by z. It is to be referred to, in the literature, as Markov-Ocneanu trace with parameter z. We consider the collection $(\tau_i)_{1\leq i}$, which is to be defined inductively by restricting τ on the tower of algebras, after specializing z to be $(\frac{P+1}{S})^{-1}$. Namely we define (for all $0 \leq n$) the functions $\tau_{n+1}(x) = (\frac{P+1}{S})^n\tau(x)$ for any $x \in H_n(q)$. By induction over n, we see that the collection $(\tau_i)_{1\leq i}$ is a Markov trace.

We have:

$$K[B(A_n)] \xrightarrow{\pi_n} H_n(q) \xrightarrow{\tau_{n+1}} K \text{ for all } 0 \leq n$$

Which gives a knot invariant for every specialization of q' and q'', in such way that P and S are invertible [GP05]. We get Alexander polynomial Δ_t, which is a Laurent polynomial in one variable $t^{\frac{1}{2}}$, by setting $q' = t^{\frac{1}{2}}$ and $q'' = -t^{\frac{1}{2}}$.

Jones polynomial is obtained by setting $q' = t^{\frac{3}{2}}$ and $q'' = -t^{\frac{1}{2}}$.

4.2 The B-type Hecke algebra

In this section we take $K = \mathbb{Q}[q^{\pm 1}, Q^{\pm 1}]$. We define a Markov trace of type B (say B-Markov trace). We give some well known results about this trace, and its relation with the Markov trace, just defined in the last subsection.

4.2.1 $H_n(q)$ as a sub-algebra of $HB_{n+1}(q)$

Lets consider the group algebra $K[B(B_{n+1})]$. Let q, Q be in K. Let IB_n be the ideal generated by the set $\{\sigma_i^2 - (q-1)\sigma_i - q, t^2 - (Q-1)t - Q;\ \ 1 \leq i \leq n\}$. We have seen that the elements $\sigma_i^2 - (q-1)\sigma_i - q$ are conjugate in $B(A_n)$ for $1 \leq i \leq n$, hence in $B(B_{n+1})$, thus, we can say that $IB_n = \langle \sigma_t^2 - (q-1)\sigma_1 - q, t^2 - (Q-1)t - Q \rangle_{K[B(B_{n+1})]}$.

We view the B-type Hecke algebra $HB_{n+1}(q, Q)$ as the quotient $K[B(B_{n+1})]/IB_n$. We set g_{σ_i} to be the image of σ_i via the natural surjection, and g_t to be the image of t (we are just about to see, why it is legitimate to choose such symbols for the images). $HB_{n+1}(q, Q)$ is presented by a set $\{g_t, g_{\sigma_i};\ \ 1 \leq i \leq n\}$ with the following relations:

- $g_{\sigma_i} g_{\sigma_j} = g_{\sigma_j} g_{\sigma_i}$, where $1 \leq i, j \leq n$ for $|i - j| \geq 2$.
- $g_{\sigma_i} g_{\sigma_{i+1}} g_{\sigma_i} = g_{\sigma_{i+1}} g_{\sigma_i} g_{\sigma_{i+1}}$, for $1 \leq i \leq n-1$.
- $g_{\sigma_i} g_t = g_t g_{\sigma_i}$, for $2 \leq i \leq n$
- $g_{\sigma_1} g_t g_{\sigma_1} g_t = g_t g_{\sigma_1} g_t g_{\sigma_1}$.
- $g_{\sigma_i}^2 = (q-1) g_{\sigma_i} + q$.
- $g_t^2 = (Q-1) g_t + Q$.

The B-type Coxeter group with $n+1$ generators $W(B_{n+1})$, indexes a K-basis of $HB_{n+1}(q, Q)$. Moreover, if we set $W(B_1) = B(B_1) = 1$, we get the following tower:

$W(B_1) \subset W(B_2) \ldots \subset W(B_n) \subset W(B_{n+1}) \subset \ldots$ which as above induces
$HB_1(q, Q) \subset HB_2(q, Q) \ldots \subset HB_n(q, Q) \subset HB_{n+1}(q, Q) \subset \ldots.$

The fact that $W(A_n)$ is a parabolic subgroup of $W(B_{n+1})$ gives that $H_n(q)$ is a parabolic subalgebra of $HB_{n+1}(q)$. We keep the same notation of 2.4.4, that is:

$$z_n : H_n(q) \longrightarrow HB_{n+1}(q).$$

Moreover, we can "shift" the K-basis, as we have done above. That is by setting $T_{\sigma_i} := -q'' g_{\sigma_i}$ for $1 \leq i \leq n$, and $T_t := g_t$. Hence we can present $HB_{n+1}(q)$ by the set of generators $\{T_t, T_{\sigma_i}; 1 \leq i \leq n\}$ with the following relations:

- $T_{\sigma_i} T_{\sigma_j} = T_{\sigma_j} T_{\sigma_i}$, where $1 \leq i, j \leq n$ for $|i - j| \geq 2$.
- $T_{\sigma_i} T_{\sigma_{i+1}} T_{\sigma_i} = T_{\sigma_{i+1}} T_{\sigma_i} T_{\sigma_{i+1}}$ for $1 \leq i \leq n - 1$.
- $T_{\sigma_i} T_t = T_t T_{\sigma_i}$ for $2 \leq i \leq n$.
- $T_{\sigma_1} T_t T_{\sigma_1} T_t = T_t T_{\sigma_1} T_t T_{\sigma_1}$.
- $T_{\sigma_i}^2 = S T_{\sigma_i} - P$.
- $T_t^2 = (Q - 1) T_t + Q$.

4.2.2 B-Markov trace

We set $HB(q, Q) := \cup_{1 \leq i} HB_i(q, Q)$. Let z, y be in K. We set

$$T_j = T_{\sigma_j} T_{\sigma_{j-1}} ... T_{\sigma_1} T_t T_{\sigma_1}^{-1} ... T_{\sigma_{j-1}}^{-1} T_{\sigma_j}^{-1} \in W(B_n+1).$$

Where $0 \leq j \leq n$ and $T_0 = t$.

Theorem 4.2.1. *[GL97] Consider the above notations. Let z, y_j be in K, for $1 \leq j$. There exists a unique trace $\tau^B : HB(q, Q) \longrightarrow$ such that:*

(1) $\tau^B(1) = 1$.

(2) $\tau^B(h e_{\sigma_n}) = z \tau^B(h)$ *for* $n \geq 1$ *and* $h \in HB_n(q, Q)$.

(3) $\tau^B(T_j) = y_j$ *for for* $1 \leq k$.

The restriction of τ^B to $H(q)$ (seen as a subalgebra of $HB(q, Q)$), is Markov trace on $H(q)$, moreover, it is the Markov trace with the very same parameter z, hence, uniquely determined. Notice that a Markov trace on $HB(q, Q)$ defines by restriction a collection of traces over $HB_n(q, Q)$ satisfying conditions, similar to those in type A.

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4.3 $\widehat{H}_{n+1}(q)$

In this section we recall the formal definitions of the extended affine Hecke algebra and the non-extended affine Hecke algebra, we list some facts about the K-basis, we show that the $\tilde{A}$-type affine Hecke algebra with n generators injects into the $\tilde{A}$-type affine Hecke algebra with $n+1$ generators. Then, we explain the surjection of the $\tilde{A}$-type affine Hecke algebra with $n+1$ generators onto the A-type Hecke algebra with n generators, which comes from the surjection β_n in 2.4.2. Until the end of this section q is an indeterminate over K.

4.3.1 Extended / Non-extended affine Hecke algebra

Consider $K[B(B_{n+1})]$, recall that $B(B_{n+1})$ is generated by S', with the relations in Proposition 2.3.2, where $S' = \{\sigma_1, \sigma_2..\sigma_n, a_{n+1}, \phi_{n+1}\}$. Let I'_n be the ideal generated by the elements $\sigma_i^2 - (q-1)\sigma_i - q$, for $1 \leq i \leq n$, and $a_{n+1}^2 - (q-1)a_{n+1} - q$, which are, as we have shown above, conjugate. The extended affine Hecke algebra associated $\overline{H}_{n+1}(q)$, is by definition, the quotient $K[B(B_{n+1})]/I'_n$. We set $g_{a_{n+1}}$ (resp. g_{σ_i} for $1 \leq i \leq n$) the image of a_{n+1} (resp. σ_i for $1 \leq i \leq n$), under the natural surjection. While the image of ϕ_{n+1} is to be denoted by ϕ_{n+1}. With ϕ_{n+1} acting on the generators by shifting, we can see directly that:

$$\overline{H}_{n+1}(q) = \left\langle g_{a_{n+1}}, g_{\sigma_i};\ \ 1 \leq i \leq n \right\rangle^{alg}_{\overline{H}_{n+1}(q)} \otimes \langle \phi_{n+1} \rangle_{\overline{H}_{n+1}(q)}.$$

We are concerned with the non-extended affine Hecke algebra, from now on, we call it the affine Hecke algebra, this is $\left\langle g_{a_{n+1}}, g_{\sigma_i};\ \ 1 \leq i \leq n \right\rangle^{alg}_{\overline{H}_{n+1}(q)}$, seen as a subalgebra of $\overline{H}_{n+1}(q)$. We denote it by $\widehat{H}_{n+1}(q)$. In details, if we set $W(\tilde{A}_0) = 1$. We see that for $0 \leq n$, the affine Hecke algebra $\widehat{H}_{n+1}(q)$ is a K-vector space with a K-basis indexed by the elements of $W(\tilde{A}_n)$, namely $\left\{g_w;\ w \in W(\tilde{A}_n)\right\}$. For w, v in $W(\tilde{A}_n)$ and s in $\{\sigma_1, .., \sigma_n, a_{n+1}\}$, the multiplication is given by:

$$\begin{aligned} g_w g_v &= g_{wv}, && \text{whenever } l(wv) = l(w) + l(v).\\ g_s g_w &= (q-1)g_w + q g_{sw}, && \text{whenever } l(sw) = l(w) - 1. \end{aligned}$$

4.3.2 $\widehat{H}_n(q)$ injects into $\widehat{H}_{n+1}(q)$

Our aim in this subsection is to show that F_n from $\widehat{H}_n(q)$ to $\widehat{H}_{n+1}(q)$, is an injection. Let $2 \leq n$. We start by the injection $F'_n : K[B(\tilde{A}_{n-1})] \longrightarrow K[B(\tilde{A}_n)]$ given by:

$$\begin{aligned} \sigma_i &\longmapsto \sigma_i \text{ for } 1 \leq i \leq n-1,\\ a_n &\longmapsto \sigma_n a_{n+1} \sigma_n^{-1}. \end{aligned}$$

Which is induced -by linearity- by the injection F_n of $B(\tilde{A}_{n-1})$ into $B(\tilde{A}_n)$ in 2.4.4. F'_n gives obviously, the next homomorphism of algebras (we call it F_n also):

$$\begin{aligned} F_n : \widehat{H}_n(q) &\longrightarrow \widehat{H}_{n+1}(q) \quad \text{given by} \\ t_i &\longmapsto g_i \text{ for } 1 \leq i \leq n-1 \\ t_{a_n} &\longmapsto g_n g_{a_{n+1}} g_n^{-1}. \end{aligned}$$

Where t_{a_n} (resp. t_{σ_i} for $1 \leq i \leq n-1$) is the image of a_n (resp. σ_i for $1 \leq i \leq n-1$), under the natural surjection of $K[B(\tilde{A}_{n-1})]$ onto $\widehat{H}_n(q)$.

And $g_{a_{n+1}}$ (resp. g_{σ_i} for $1 \leq i \leq n$) is the image of a_n (resp. σ_i for $1 \leq i \leq n-1$), under the natural surjection of $K[B(\tilde{A}_n)]$ onto $\widehat{H}_{n+1}(q)$.

In order to simplify, we set $F = F_n$ until the end of this section.

We consider the injection:

$$\begin{aligned} I : W(\tilde{A}_{n-1}) &\longrightarrow W(\tilde{A}_n) \text{ given by} \\ \sigma_i &\longmapsto \sigma_i \text{ for } 1 \leq i \leq n-1 \\ a_n &\longmapsto \sigma_n a_{n+1} \sigma_n. \end{aligned}$$

Which gives, by linearity, the following algebra monomorphism (keeping the same notation for the two monomorphisms):

$$\begin{aligned} I : K[W(\tilde{A}_{n-1})] &\longrightarrow K[W(\tilde{A}_n)] \text{ given by} \\ \sigma_i &\longmapsto \sigma_i \text{ for } 1 \leq i \leq n-1 \\ a_n &\longmapsto \sigma_n a_{n+1} \sigma_n. \end{aligned}$$

Now, we get the next diagram:

$$\begin{tikzcd} \widehat{H}_n(q) \arrow[r, "F"] \arrow[d, two heads, "M_n"'] & \widehat{H}_{n+1}(q) \arrow[d, two heads, "M_{n+1}"] \\ K[W(\tilde{A}_{n-1})] \arrow[r, hook, "I"'] & K[W(\tilde{A}_n)] \end{tikzcd}$$

Figure 4.1: *DM*

Where M_n and M_{n+1} are the maps coming from specializing q to 1.

Lemma 4.3.1. *Let w be any element in $W(\tilde{A_{n-1}})$. Then:*

$$F(t_w) = Ag_{I(w)} + (q-1) \sum_{x \in W(\tilde{A_n})} \lambda_x g_x.$$

Where A and λ_x are polynomials in q and q^{-1} over $\mathbb{Q}$ with $A(1) = 1$.

Proof. Suppose $l(w) = 1$, if w is in $\{\sigma_1, \sigma_2, \dots \sigma_{n-1}\}$, then:

$$F(t_w) = F(t_i) = g_i \text{ for } 1 \leq i \leq n-1.$$

$$\text{Moreover, } F(t_w) = g_{\sigma_n} g_{a_{n+1}} g_{\sigma_n}^{-1} = \frac{1}{q} g_{\sigma_n} g_{a_{n+1}} g_{\sigma_n} + \frac{1-q}{q} g_{\sigma_n} g_{a_{n+1}}.$$

Now take w, with $2 \leq l(w)$, suppose that statement is true for any word of length h where $h < l(w)$. Again, if $w \in W(A_{n-1})$, then $F(t_w) = g_w$, keeping in mind that in this case $I(w) = w$, hence, our statement in true in this case. Suppose that a_n appears in one (hence every) reduced expression of w, then w could be written $w = ua_nv$ where v is a word in $\sigma_1, \sigma_2, \dots \sigma_{n-1}$, and such that $l(w) = l(u) + l(v) + 1$.

$$F(t_w) = F(t_u)F(t_{a_n})F(t_v) = F(t_u) g_{\sigma_n} g_{a_{n+1}} g_{\sigma_n}^{-1} g_v.$$

By the induction hypothesis, since $l(u) \leq k-1$, we can write $F(t_u)$ as follows:

$$F(t_u) = Ag_{I(u)} + (q-1) \sum_{y \in W(\tilde{A_n})} \mu_y g_y. \text{ Hence,}$$

$$F(t_w) = \frac{1}{q}\left(Ag_{I(u)} + (q-1) \sum_{y \in W(\tilde{A_n})} \mu_y g_{a_{n+1}} \right) g_{\sigma_n} g_{a_{n+1}} g_{\sigma_n} g_v +$$

$$(q-1)\ q\left(Ag_{I(u)} + (q-1) \sum_{y \in W(\tilde{A_{n-1}})} \mu_y g_y \right) g_{\sigma_n} g_{a_{n+1}} g_{\sigma_n}.$$

By expanding we see that all the terms have the form $(q-1)\mu_y g_y$ where as said before, μ_y is in $\mathbb{Q}[q, q^{-1}]$ and y is in $W(\tilde{A_n})$, hence,

$$F(t_w) = \frac{1}{q} Ag_{I(u)} g_{\sigma_n} g_{a_{n+1}} g_{\sigma_n} g_v + (q-1) \sum_{y \in W(\tilde{A_n})} \mu_y g_y.$$

Now, $g_{I(u)}g_{\sigma_n} = (q-1)g_{I(u)} + qg_{I(u)\sigma_n}$. Hence,

$$\begin{aligned} g_{I(u)}g_{\sigma_n}g_{a_{n+1}} &= (q-1)g_{I(u)}g_{a_{n+1}} + qg_{I(u)\sigma_n}g_{a_{n+1}} \\ &= (q-1)g_{I(u)}g_{a_{n+1}} + q\Big((q-1)g_{I(u)\sigma_n} + qg_{I(u)\sigma_n a_{n+1}}\Big) \\ &= (q-1)\Big(g_{I(u)}g_{a_{n+1}} + qg_{I(u)\sigma_n}\Big) + q^2 g_{I(u)\sigma_n a_{n+1}}. \end{aligned}$$

Hence, $g_{I(u)}g_{\sigma_n}g_{a_{n+1}}g_{\sigma_n} = (q-1)(g_{I(u)}g_{a_{n+1}} + qg_{I(u)}\sigma_n)g_{\sigma_n} + q^2 g_{I(u)\sigma_{\sigma_n}a_{n+1}}g_{\sigma_n}$.

But $q^2 g_{I(u)\sigma_{\sigma_n}a_{n+1}}g_n = q^2(q-1)g_{I(u)\sigma_n a_{n+1}} + q^3 g_{I(u)\sigma_n a_{n+1}\sigma_n}$.

Thus,

$$g_{I(u)}g_n g_{a_{n+1}}g_n = (q-1)\underbrace{\left(g_{I(u)}g_{a_{n+1}}g_n + qg_{I(u)}\sigma_n g_n + q^2 g_{I(u)\sigma_n a_{n+1}}\right)}_{E} + q^3 g_{I(u)\sigma_n a_{n+1}\sigma_n}.$$

But, we have $q^3 g_{I(u)\sigma_n a_{n+1}\sigma_n} = q^3 g_{I(u)} I(a_n) = q^3 g_{I(ua_n)}$.

Now, we have $F(t_w) = 1/qA((q-1)E + q^3 g_{I(ua_n)})g_v + (q-1)\sum_{y\in W(\tilde{A_n})} \mu_y g_y$.

Which is equal to:

$q^2 g_{I(u)a_n}g_v + (q-1)\sum_{x\in W(\tilde{A_n})} \mu_x g_x$. Since and for $l(w) = l(u) + l(a_n) + l(v)$.

We see that $F(t_w) = A' g_{I(w)} + (q-1)\sum_{x\in W(\tilde{A_n})} \lambda_x g_x$.

Where A' is to be $q^2 A$, our statement is proved to be true.

□

Corollary 4.3.2. *The diagram DM is commutative.*

Proof. The diagram commutes, if and only if, for each w in $W(\tilde{A}_n)$, we have:

$$I\Big(M_n(t_w)\Big) = M_{n+1}\Big(F(t_w)\Big). \tag{4.1}$$

Now $I\Big(M_n(t_w)\Big) = I(w)$, while, by lemma 4.3.1, we see that $M_{n+1}(F(t_w))$ is equal to:

$$M_{n+1}\Big(A'g_{I(w)} + (q-1)\sum_{x\in W(\tilde{A_n})} \lambda_x g_x\Big) = A'(1)M_{n+1}(g_{I(w)}).$$

Finally, we get:

$$M_{n+1}\Big(F(t_w)\Big) = M_{n+1}\Big(g(I_w)\Big) = I(w).$$

□

Proposition 4.3.3. * *The following homomorphism of algebras is a monomorphism:*

$$\begin{aligned} F : \widehat{H}_n(q) &\longrightarrow \widehat{H}_{n+1}(q) \\ t_i &\longmapsto g_i \textit{ for } 1 \le i \le n-1 \\ t_{a_n} &\longmapsto g_n g_{a_{n+1}} g_n^{-1}. \end{aligned}$$

Proof. We will make use of the fact that the diagram DM commutes, now, $\widehat{H}_n(q)$ surjects onto $Im(F)$ by definition, we prove that the images of the basis of $\widehat{H}_n(q)$ are linearly independent in $\widehat{H}_{n+1}(q)$.

Suppose that the statement is not true, i.e., suppose that there exist polynomials in $\mathbb{Q}[q,q^{-1}]$, such as λ_w, which are not all zero, with $\sum_{w\in W(A_{\tilde{n-1}})} \lambda_w F(t_w) = 0$.

Now for each w, we write $F(t_w)$ expressed by the elements of the basis of $\widehat{H}_n(q)$:

$$F(t_w) = \sum_{v\in W(\tilde{A_n})} \mu_v^w g_v. \text{ Thus } M_{n+1}(F(t_w)) = \sum_{v\in W(\tilde{A_n})} \mu_v^w(1)v.$$

But, since the diagram commutes, we have:

$$M_{n+1}\Big(F(t_w)\Big) = I\Big(M_{n+1}(t_w)\Big) = I(w).$$

$$\text{Now, } \sum_{w\in W(A_{\tilde{n-1}})} \lambda_w F(t_w) = 0.$$

We apply M_{n+1} to get:

$$\sum_{w\in W(A_{\tilde{n-1}})} \lambda_w(1)I(w) = I\Big(\sum_{w\in W(A_{\tilde{n-1}})} \lambda_w(1)w\Big) = 0, \text{ thus } \sum_{w\in W(A_{\tilde{n-1}})} \lambda_w(1)w = 0.$$

Which implies that $\lambda_w(1) = 0$, for every w.

This last fact means that the polynomial $(q-1)$ divides every λ_w, so every λ_w can be written as a product of $(1-q)^r$ (for some positive r) by a polynomial which does not vanish for $q=1$, we can choose the λ_w with r minimum. factorizing by $(1-q)^r$ in $\sum_{w \in W(\tilde{A_{n-1}})} \lambda_w(1)w = 0$, we can find, at least, a non zero polynomial still verifying the last quality, which contradicts the fact that the elements of the group are linearly independent in the group algebra, so the hypothesis in not valid and F is injection.

$\square$

Remark 4.3.4. *After showing that F_n is injective, we can denote the generators of $\widehat{H}_n(q)$ by $g_{\sigma_1}, ..., g_{\sigma_{n-1}}, g_{a_n}$ and the generators of $\widehat{H}_{n+1}(q)$ by $g_{\sigma_1}, ..., g_{\sigma_n}, g_{a_{n+1}}$, hence F_n takes the form:*

$$\begin{aligned} F_n : \widehat{H}_n(q) &\longrightarrow \widehat{H}_{n+1}(q) \\ g_{\sigma_i} &\longmapsto g_{\sigma_i} \textit{ for } 1 \leq i \leq n-1 \\ g_{a_n} &\longmapsto g_{\sigma_n} g_{a_{n+1}} g_{\sigma_n}^{-1}. \end{aligned}$$

4.3.3 $\widehat{H}_{n+1}(q)$ surjects onto $H_n(q)$

Let K be a field, recall the injection:

$$\begin{aligned} F_n : \widehat{H}_n(q) &\longrightarrow \widehat{H}_{n+1}(q) \\ g_{\sigma_i} &\longmapsto g_{\sigma_i} \text{ for } 1 \leq i \leq n-1 \\ g_{a_n} &\longmapsto g_{\sigma_n} g_{a_{n+1}} g_{\sigma_n}^{-1}. \end{aligned}$$

We consider the element $g_{\sigma_1\sigma_2\ldots\sigma_n} - g_{a_{n+1}\sigma_1\ldots\sigma_{n-1}}$ in $\widehat{H}_{n+1}(q)$, for $n \geq 2$, (for $n=2$ it is to be $g_{\sigma_1} - g_{a_2}$). We set $N_n := \left\langle g_{\sigma_1\sigma_2\ldots\sigma_n} - g_{a_{n+1}\sigma_1\ldots\sigma_{n-1}} \right\rangle_{\widehat{H}_{n+1}(q)}$, that is the ideal generated by this element in $\widehat{H}_{n+1}(q)$. Let M_n be the quotient by this ideal, i.e., $M_n := \widehat{H}_{n+1}(q)/N_n$.

Keeping the same notations for the generators of $\widehat{H}_n(q)$(resp. $\widehat{H}_{n+1}(q)$) and the images under the natural surjection in M_{n-1} (resp. M_n), we claim that F_n induces an algebra homomorphism m:

$$\begin{aligned} m : M_{n-1} &\longrightarrow M_n \text{ given by:} \\ g_{\sigma_i} &\longmapsto g_{\sigma_i} \text{ for } 1 \leq i \leq n-1 \\ g_{a_n} &\longmapsto g_{\sigma_n} g_{a_{n+1}} g_{\sigma_n}^{-1}. \end{aligned}$$

Remark 4.3.5. *We will see in what follows that m is injective, that is why it is legitimate to keep the same symbols for the generators.*

To make sure that m is indeed a homomorphism, we are reduced to show that:

$m(g_{\sigma_1\sigma_2\ldots\sigma_{n-1}} - g_{a_n\sigma_1\ldots\sigma_{n-2}}) = 0,$ which can be done as follows:

$m(g_{\sigma_1\sigma_2\ldots\sigma_{n-1}} - g_{a_n\sigma_1\ldots\sigma_{n-2}}) = g_{\sigma_1\sigma_2\ldots\sigma_{n-1}} - g_{\sigma_n}g_{a_{n+1}}g_{\sigma_n}^{-1}g_{\sigma_1\ldots\sigma_{n-2}}$. Which is:

$$g_{\sigma_1\sigma_2\ldots\sigma_{n-1}} - g_{\sigma_n a_{n+1}\sigma_1\ldots\sigma_{n-2}}g_{\sigma_n}^{-1} = g_{\sigma_1\sigma_2\ldots\sigma_{n-1}} - g_{\sigma_n}g_{\sigma_1..\sigma_{n-1}}g_{\sigma_n}g_{\sigma_{n-1}}^{-1}..g_{\sigma_1}^{-1}g_{\sigma_1\ldots\sigma_{n-2}}g_{\sigma_n}^{-1}.$$

Now, $g_{\sigma_1\sigma_2\ldots\sigma_{n-1}} - g_{\sigma_n\sigma_1..\sigma_{n-1}\sigma_n}g_{\sigma_{n-1}}^{-1}g_{\sigma_1}^{-1} = g_{\sigma_1\sigma_2\ldots\sigma_{n-1}} - g_{\sigma_1}..\underbrace{g_{\sigma_n\sigma_{n-1}\sigma_n}}_{g_{\sigma_{n-1}\sigma_n\sigma_{n-1}}}g_{\sigma_{n-1}}^{-1}g_{\sigma_n}^{-1}$.

Hence, it is equal to $g_{\sigma_1\sigma_2\ldots\sigma_{n-1}} - g_{\sigma_1\sigma_2\ldots\sigma_{n-1}} = 0$.

So, m is compatible with the injection F_n. We have the following map:

$$\begin{aligned} r' : H_n(q) &\longrightarrow M_n \\ g_{\sigma_i} &\longmapsto g_{\sigma_i} \text{ for } 1 \leq i \leq n, \end{aligned}$$

which is obviously a homomorphism. Moreover, it is surjective, since $g_{a_{n+1}}$ could be considered as $r'(g_{\sigma_1}..g_{\sigma_{n-1}}g_{\sigma_n}g_{\sigma_{n-1}}^{-1}..g_{\sigma_1}^{-1})$. So M_n is linearly finitely generated, with:

$$dim(M_n) \leq dim\Big(H_n(q)\Big).$$

Now let r be the map:

$$\begin{aligned} r : M_n &\longrightarrow H_n(q) \\ g_{\sigma_i} &\longmapsto g_{\sigma_i} \text{ for } 1 \leq i \leq n \\ g_{a_{n+1}} &\longmapsto g_{\sigma_1}..g_{\sigma_{n-1}}g_{\sigma_n}g_{\sigma_{n-1}}^{-1}..g_{\sigma_1}^{-1}. \end{aligned}$$

This map is an algebra homomorphism if the following statements are true:

1. $g_{\sigma_i}r(g_{a_{n+1}}) = r(g_{a_{n+1}})g_{\sigma_i}$ when $2 \leq i \leq n-1$.
2. $g_{\sigma_1}r(g_{a_{n+1}})g_{\sigma_1} = r(g_{a_{n+1}})g_{\sigma_1}r(g_{a_{n+1}})$.
3. $g_{\sigma_n}r(g_{a_{n+1}})g_{\sigma_n} = r(g_{a_{n+1}})g_{\sigma_n}r(g_{a_{n+1}})$.
4. $(r(g_{a_{n+1}}))^2 = (q-1)r(g_{a_{n+1}}) + q$.

These relations can easily shown to be true, by direct computation, so we have an algebra isomorphism $M_n \xrightarrow{\sim} H_n(q)$. That gives rise to a new point of view of $\widehat{H}_{n+1}(q)$, that is:

$$\widehat{H}_{n+1}(q) = H_n(q) \oplus N_n.$$

Now, notice that $\sigma_1\sigma_2...\sigma_n - a_{n+1}\sigma_1...\sigma_{n-1}$ in $K[B(\tilde{A}_n)]$ is sent to $g_{\sigma_1}g_{\sigma_2}...g_{\sigma_n} - g_{a_{n+1}}g_{\sigma_1}...g_{\sigma_{n-1}}$, in $\widehat{H}_{n+1}(q)$ by the natural surjection. We set N'_n to be the ideal generated by $\sigma_1\sigma_2...\sigma_n - a_{n+1}\sigma_1...\sigma_{n-1}$ in $K[B(\tilde{A}_n)]$. We see that Hecke quadratic relations are respected by the surjection of $K[B(\tilde{A}_n)]$ onto $K[B(A_n)]$, in other terms we have:

$$\begin{array}{ccccc}
K[B(\tilde{A}_n)] & = & K[B(A_n)] & \oplus & N'_n \\
\downarrow & & \downarrow{\scriptstyle \pi'_n} & & \downarrow \\
\widehat{H}_{n+1}(q) & = & H_n(q) & \oplus & N_n
\end{array}$$

4.4 Bibliographical remarks and problems

In the proof of proposition 4.3.3, the idea of making use of the fact that the diagram DM is commutative, is due to a discussion with F. Digne and J. Michel. Obviously, $\widehat{H}_{n-1}(q)$ is not parabolic in $\widehat{H}_n(q)$ and both are of infinite dimensions, hence, we do not have the tools of the classical theory. For instance, those tools (in particular, irreducible characters of Hecke algebras and elements of minimal length in the conjugacy classes) were the keys of the classification of all traces over different Hecke algebras of spherical type, in [GP93], of Geck and Pfeiffer. Ten years later, in in [GL97], Geck and Lambropoulou parametrized all Markov traces (among those of the classification of [GP93]), that is the B-Markov trace of theorem 4.2.1.

It is well known that Hecke algebra associated to some spherical Coxeter group is isomorphic to the group algebra of this Coxeter group, when the ground field K is algebraically closed (Tit's deformation, [Bou81]). "Generic" cases were treated by Lusztig and Geck (equal and nonequal parameters) [Gec11].

Unfortunately, we are far from finding an affine version of this isomorphism, i.e., an isomorphism between $\widehat{H}_{n+1}(q)$ and $K[W(\tilde{A}_n)]$, at least, not by using the classical theory, the problem still stands.

قالتْ فإنّ القصرَ من دوننا, قلتُ فإنّي فوقه ظاهرْ. وضّاح اليمنْ

Elle a dit : il s'agit d'une tour, une énorme tour.

Très bien ... j'arrive, ai-je répondu.

Chapter

5 Temperley-Lieb algebras

Let (W, S) be a Coxeter system, let $H(W)$ be the associated Hecke algebra. let $J(W)$ be the ideal of $H(W)$ generated by the elements $\sum_{w \in \langle s,t \rangle} g_w$, where (s, t) runs over all pairs of elements of S that correspond to adjacent nods in Γ $\Big($the Dynkin graph associated to $(W, S)\Big)$ such that $m_{s,t} < \infty$.

Definition 5.0.1. * *The generalized Temperley-Lieb algebra $TL(W)$ is the quotient $H(W)/J(W)$. We denote the natural surjection $\theta : H(W) \longrightarrow TL(W)$. We call it T-L algebra.*

Let v be in W^c. We denote $\theta(g_u)$ by g_u in $TL(W)$, where in $J(W)$: g_u is equal to $\theta(g_{s_1} g_{s_2} .. g_{s_r}) = g_{s_1} g_{s_2} .. g_{s_r}$, where $s_1 s_2 .. s_r$ is some reduced expression of u. This notation makes sense thanks to the following proposition.

Proposition 5.0.2. * *The set $\{g_w : w \in W^c\}$ is a K-basis of $TL(W)$, where $g_w := g_{s_1} g_{s_2} .. g_{s_r}$ for any fully commutative element w, and any reduced expression of $w = s_1 s_2 .. s_r$.*

for x, y in $H(W)$ (resp. $J(W)$), we define the function V as follows:

$V : H(W)$ (resp. $J(W)) \longrightarrow H(W)$ (resp. $J(W)$)

$$V(x, y) = xyx + xy + yx + x + y + 1.$$

5.1 A-type Temperley-Lieb algebras *

Here we take $K = \mathbb{Q}[\sqrt{q}]$, here q is an indeterminate.

The A-type T-L algebra with n generators $TL_n(q)$ has a proper sense when $2 \leq n$. We put $TL_1(q) := H_1$. For $2 \leq n$, we present $TL_n(q)$ by a set of generators $\{g_{\sigma_1}, \dots, g_{\sigma_n}\}$, with the following defining relations:

- $g_{\sigma_i} g_{\sigma_j} = g_{\sigma_j} g_{\sigma_i}$, for $1 \leq i, j \leq n$ and $|i-j| \geq 2$.
- $g_{\sigma_i} g_{\sigma_{i+1}} g_{\sigma_i} = g_{\sigma_{i+1}} g_{\sigma_i} g_{\sigma_{i+1}}$, for $1 \leq i \leq n-1$.
- $g_{\sigma_i}^2 = (q-1) g_{\sigma_i} + q$, for $1 \leq i \leq n$.
- $V(g_{\sigma_i}, g_{\sigma_{i+1}}) = 0$, for $1 \leq i \leq n-1$.

As said above $TL_n(q)$ has a canonical K-basis $\{g_w : w \in W^c(A_n)\}$.

5.1.1 Presentations

We set $\delta = \dfrac{1}{2+q+q^{-1}} = \dfrac{q}{(1+q)^2}$. Set $f_{\sigma_i} = \dfrac{g_{\sigma_i}+1}{q+1}$ with for $i = 1, 2, \dots, n$.

Hence, $g_{\sigma_i} = (q+1) f_{\sigma_i} - 1$ for $i = 1, 2, \dots, n$.

Proposition 5.1.1. *The algebra $TL_{n+1}(q)$ is presented by the set $\{f_{\sigma_i} : 1 \leq i \leq n\}$, with the following defining relations:*

- $f_{\sigma_i} f_{\sigma_j} = f_{\sigma_j} f_{\sigma_i}$, *for $1 \leq i, j \leq n$ and $|i-j| \geq 2$.*
- $f_{\sigma_i}^2 = f_{\sigma_i}$, *for all i.*
- $f_{\sigma_i} f_{\sigma_{i+1}} f_{\sigma_i} = \delta f_{\sigma_i}$, *for $1 \leq i \leq n-1$.*
- $f_{\sigma_i} f_{\sigma_{i-1}} f_{\sigma_i} = \delta f_{\sigma_i}$, *for $2 \leq i \leq n$.*

Moreover, f_u is well defined in the natural way, and the set $\{f_w : w \in W^c(A_n)\}$ is a K-basis of $TL_n(q)$.

Now, we set $E_i := \dfrac{1+q}{\sqrt{q}} f_i = \dfrac{1}{\sqrt{\delta}} f_i$.

Proposition 5.1.2. *The algebra $TL_{n+1}(q)$ is presented by the set $\{E_{\sigma_i} : 1 \leq i \leq n\}$, with the following defining relations:*

- $E_{\sigma_i} E_{\sigma_j} = E_{\sigma_j} E_{\sigma_i}$ *for $1 \leq i, j \leq n$ and $|i-j| \geq 2$.*
- $E_{\sigma_i}^2 = \frac{1}{\sqrt{\delta}} E_{\sigma_i}$ *for all i.*
- $E_{\sigma_i} E_{\sigma_{i+1}} E_{\sigma_i} = E_{\sigma_i}$. *for $1 \leq i \leq n-1$.*
- $E_{\sigma_i} E_{\sigma_{i-1}} E_{\sigma_i} = E_{\sigma_i}$. *for $2 \leq i \leq n$.*

E_u is well defined in the natural way, and the set $\{E_w : w \in W^c(A_n)\}$ is a K-basis.

This presentation is called diagrammatic presentation, for, each E_{σ_j} can be presented as a diagram, it will be explained in the next section.

5.1.2 τ_{n+1} factors through $TL_n(q)$

We set θ_n the natural surjection $H_n(q) \longrightarrow TL_n(q)$ (θ_1 is the Id map). Basically, the ideal by which we quotient $H_n(q)$ is the ideal generated by $\sum\limits_{w\in\langle\sigma_i,\sigma_{i+1}\rangle} g_w$ for $1 \leq i \leq n-1$, i.e., the ideal generated by $V(g_{\sigma_i}, g_{\sigma_{i+1}})$, for $1 \leq i \leq n-1$. If $n \leq 3$ we see that

$$g_{\sigma_i} g_{\sigma_{i+1}} g_{\sigma_{i+2}} V(g_{\sigma_i}, g_{\sigma_{i+1}}) g_{\sigma_{i+2}}^{-1} g_{\sigma_{i+1}}^{-1} g_{\sigma_i}^{-1} = g_{\sigma_i\sigma_{i+1}\sigma_{i+2}} V(g_{\sigma_i}, g_{\sigma_{i+1}}) g_{\sigma_i\sigma_{i+1}\sigma_{i+2}}^{-1},$$

which is equal to $V(g_{\sigma_{i+1}}, g_{\sigma_{i+2}})$. In other terms: the ideal $J(A_n)$ is $\langle V(g_{\sigma_1}, g_{\sigma_2})\rangle_{H_n(q)}$.

We consider the Markov trace $(\tau_i)_{1\leq i}$ on the tower:

$$H_0(q) \ \subset H_1(q) \ \ ... \subset H_{n-1}(q) \quad \subset H_n(q) \ \ ...$$

This tower induces obviously, a T-L algebras tower, the fact that $TL_n(q)$ could be viewed as a vector subspace of $H_n(q)$, together with the parabolicity of $H_{n-1}(q)$ in $H_n(q)$, allows to view the TL-tower as a tower of inclusions, as follows:

$$TL_0(q) \ \subset TL_1(q) \ \ ... \subset TL_{n-1}(q) \quad \subset TL_n(q) \ \ ...$$

In $H_2(q)$, we consider the element $V(g_{\sigma_1}, g_{\sigma_2})$. We apply τ_3 to this element:

$$\begin{aligned}
\tau_3\Big(V(g_{\sigma_1}, g_{\sigma_2})\Big) &= \tau_3\Big(g_{\sigma_2} g_{\sigma_1} g_{\sigma_2} + g_{\sigma_2} g_{\sigma_1} + g_{\sigma_1} g_{\sigma_2} + g_{\sigma_2} + g_{\sigma_1} + 1\Big) \\
&= \Big(\frac{-1}{q''}\Big)^3 \tau_3\Big(T_{\sigma_2} T_{\sigma_1} T_{\sigma_2}\Big) + 2\Big(\frac{-1}{q''}\Big)^2 \tau_3\Big(T_{\sigma_2} T_{\sigma_1}\Big) + \frac{-1}{q''}\tau_3\Big(T_{\sigma_2}\Big) \\
&\quad + \frac{-1}{q''}\tau_3\Big(T_{\sigma_1}\Big) + \tau_3\Big(1\Big).
\end{aligned}$$

$$\text{That is: } \tau_3\Big(V(g_{\sigma_1}, g_{\sigma_2})\Big) = \frac{q'^2(q''^2-1)(q'+q''^3)}{q''^3 S^2}.$$

So $\tau_3\Big(V(g_{\sigma_1}, g_{\sigma_2})\Big) = 0$ when $q'' = \pm 1$ or $q' = -q''^3$. We are interested in the case where q'' is not a constant, that is $q' = -q''^3$, That is $q'' = -q^{\frac{1}{2}}$ and $q' = q^{\frac{3}{2}}$ (modulo a change of variable). Notice that this is the case of Jones polynomial (in the literature q is denoted by t, as in last section). From now on, we keep this notation. Here, $S = q' + q'' = \sqrt{q}(q-1)$ and $P = q'q'' = -q^2$, while $\frac{P+1}{S} = -\frac{1+q}{\sqrt{q}}$. Since we are interested in T-L algebra until the end of this work. The reason of such choice is the following proposition.

Proposition 5.1.3. *With the above notation, we have:*

- *For $2 \leq n$, the trace τ_{n+1} is null on $\langle V(g_{\sigma_1}, g_{\sigma_2})\rangle_{H_n(q)}$.*
- *For $0 \leq n$, the trace τ_{n+1} on $H_n(q)$ factorizes through $TL_n(q)$. We keep the same notation τ_{n+1}, to denote the trace induced on $TL_n(q)$.*

Proof. Using a direct computation the proof follows. □

Hence, there exists a unique collection of traces $(\tau_{n+1})_{0\leq n}$ over $(TL_n)_{0\leq n}$, such that:

(1) $\tau_1(1) = 1$.

(2) For $1 \leq n$, we have $\tau_{n+1}(hT_{\sigma_n}^{\pm 1}) = \tau_n(h)$, for any h in $TL_{n-1}(q)$.

(3) For $1 \leq n$, every $\tau_{n+1} : TL_n(q) \longrightarrow K$ verifies

$\tau_{n+1}(bT_{\sigma_n}c) = \tau_n(bc)$ and $\tau_{n+1}(a) = -\frac{1+q}{\sqrt{q}}\tau_n(a)$. As above T_{σ_n} here is $\sqrt{q}g_{\sigma_n}$.

5.2 $\widehat{TL}_{n+1}(q)$

Let $2 \leq n$. We define $\widehat{TL}_{n+1}(q)$ to be the quotient of $\widehat{H}_{n+1}(q)$ by $\langle V(g_{\sigma_1}, g_{\sigma_2})\rangle_{\widehat{H}_{n+1}(q)}$. We have a presentation of $\widehat{TL}_{n+1}(q)$ [GL98], with set of generators $\left\{g_{\sigma_1}..g_{\sigma_n}, g_{a_{n+1}}\right\}$, and with the following relations:

- $g_{\sigma_i}g_{\sigma_j} = g_{\sigma_j}g_{\sigma_i}$, for $1 \leq i, j \leq n$ and $|i-j| \geq 2$.
- $g_{\sigma_i}g_{a_{n+1}} = g_{a_{n+1}}g_{\sigma_i}$, for $2 \leq i \leq n-1$ and $|i-j| \geq 2$.
- $g_{\sigma_i}g_{\sigma_{i+1}}g_{\sigma_i} = g_{\sigma_{i+1}}g_{\sigma_i}g_{\sigma_{i+1}}$, for $1 \leq i \leq n-1$.
- $g_{\sigma_i}g_{a_{n+1}}g_{\sigma_i} = g_{a_{n+1}}g_{\sigma_i}g_{a_{n+1}}$, for $1 = 1, n$.
- $g_{\sigma_i}^2 = (q-1)g_{\sigma_i} + q$, for $1 \leq i \leq n$.
- $g_{a_{n+1}}^2 = (q-1)g_{a_{n+1}} + q$,
- $V(g_{\sigma_i}, g_{\sigma_{i+1}}) = V(g_{\sigma_1}, g_{a_{n+1}}) = V(g_{\sigma_n}, g_{a_{n+1}}) = 0$, for $1 \leq i \leq n-1$.

The set $\left\{g_w : w \in W^c(\tilde{A}_n)\right\}$ is a K-basis. As above, we make a base change by setting $T_{a_{n+1}}$ (resp T_{σ_i} for $1 \leq i \leq n$) to be $\sqrt{q}g_{a_{n+1}}$ (resp $\sqrt{q}g_{\sigma_i}$ for $1 \leq i \leq n$). Hence, T_w is well defined for $w \in W^c(\tilde{A}_n)$, and it is equal to $q^{\frac{l(w)}{2}}g_w$. The multiplication associated to the basis $\left\{T_w : w \in W^c(\tilde{A}_n)\right\}$, is given as follows:

$$T_w T_v = T_{wv} \qquad \text{whenever } l(wv) = l(w) + l(v).$$
$$T_s T_w = \sqrt{q}(q-1)T_w + q^2 T_{sw} \qquad \text{whenever } l(sw) = l(w) - 1.$$

For w, v in $W^c(\tilde{A}_n)$ and s in $\{\sigma_1, ..., \sigma_n, a_{n+1}\}$.

5.2.1 Presentations

The algebra $\widehat{TL}_{n+1}(q)$ has two presentations [GL03] [GL98], similar to those of $TL_{n+1}(q)$ in 5.1.1. In particular $\widehat{TL}_{n+1}(q)$ can be presented by the set $\left\{E_{a_{n+1}}, E_{\sigma_i};\ \ 1 \leq i \leq n\right\}$, with the relations in proposition 5.1.2 together with the following relations:

- $E_{a_{n+1}} E_{\sigma_j} = E_{a_{n+1}} E_{\sigma_i}$, for $2 \leq i \leq n-1$.
- $E_{\sigma_i} E_{a_{n+1}} E_{\sigma_i} = E_{\sigma_i}$, for $i = 1, n$.
- $E_{a_{n+1}} E_{\sigma_i} E_{a_{n+1}} = E_{a_{n+1}}$, for $i = 1, n$.
- $E^2_{a_{n+1}} = \frac{1}{\sqrt{\delta}} E_{a_{n+1}}$.

Where E_{σ_i} (resp. $E_{a_{n+1}}$) to be $\frac{1}{\sqrt{\delta}}\frac{g_{\sigma_i}+1}{q+1}$ (resp. $\frac{1}{\sqrt{\delta}}\frac{g_{a_{n+1}}+1}{q+1}$). This is called the diagrammatic presentation. The generators can be viewed as follows:

Figure 5.1: E_1, E_{σ_1}

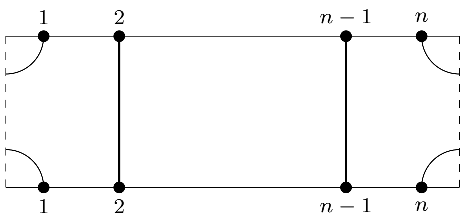


Figure 5.2: $E_{a_{n+1}}$

The diagrams are multiplied by concatenation in the usual way. E_{σ_i} for $1 \leq i \leq n$ generate $TL_{n+1}(q)$. It was shown by Jones that this presentation is faithful. The same thing is true for $\widehat{TL}_{n+1}(q)$. Saying that $\widehat{TL}_{n+1}(q)$ is given by the (generators and relations)-presentation above, is equivalent to saying that: its diagrammatic presentation is faithful, and it was proved to be so, in [FG99]. The diagrams resulting from the concatenation of $E_{a_{n+1}}$ and E_{σ_i} are called, in this work: affine diagrams (in literature: admissible affine diagrams). As shown in the following diagram, they are such that the boundary points are in even number:

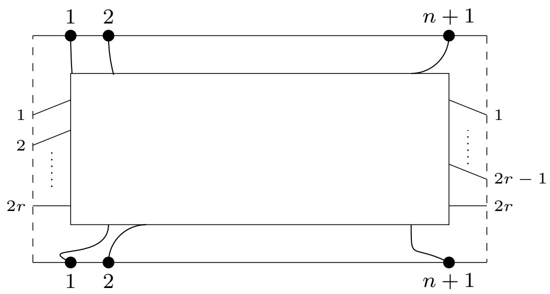


Figure 5.3: An affine diagram

5.2.2 $\widehat{TL}_{n+1}(q)$ surjects onto $TL_n(q)$

We recall the surjection of 4.3.3 from $\widehat{H}_{n+1}(q)$ onto $H_n(q)$. We call it e_n. We claim that this surjection induces a surjection from $\widehat{TL}_{n+1}(q)$ onto $TL_n(q)$, in the same way as for the Hecke algebras, i.e., we consider the set $\left\{g_{\sigma_1}, g_{\sigma_2}, ..., \; g_{\sigma_n}, g_{a_{n+1}}\right\}$ as the usual set of generators of $\widehat{H}_{n+1}(q)$ (we keep the same symbols for the images of generators under the natural surjection of $\widehat{H}_{n+1}(q)$ onto $\widehat{TL}_{n+1}(q)$, called θ'_n). Now e_n sends g_{σ_i} to g_{σ_i} in $H_n(q)$ for $1 \leq i \leq n$, while it sends $g_{a_{n+1}}$ to $g_{\sigma_1}..g_{\sigma_{n-1}}g_{\sigma_n}g_{\sigma_{n-1}}^{-1}..g_{\sigma_1}^{-1}$. We define e'_n from $\widehat{TL}_{n+1}(q)$ to $TL_n(q)$ in the very same way. It is clear that to show that e'_n is a homomorphism, we are reduced to show that in $TL_n(q)$, we have:

$$V\Big(e'_n(g_{a_{n+1}}), e'_n(g_{\sigma_1})\Big) = V\Big(e'_n(g_{\sigma_n}), e'_n(g_{a_{n+1}})\Big) = 0,$$

with $e'_n(g_{a_{n+1}}) = e_n\theta_n(g_{a_{n+1}})$ and $V\Big(e'_n(g_{a_{n+1}}), e'_n(g_{\sigma_1})\Big) = e_n\theta'_n\Big(V(g_{a_{n+1}}, g_{\sigma_1})\Big)$.

As we mentioned in 5.1.2, we see that $V(g_{a_{n+1}}, g_{\sigma_1}) = g_{a_{n+1}\sigma_1\sigma_2}^{-1}V(g_{\sigma_1}, g_{\sigma_2})g_{a_{n+1}\sigma_1\sigma_2}$. The same for $V(g_{\sigma_n}, g_{a_{n+1}})$, which is equal to $g_{\sigma_n a_{n+1}\sigma_1}^{-1}V(g_{\sigma_1}, g_{\sigma_2})g_{\sigma_n a_{n+1}\sigma_1}$. Since $V(g_{\sigma_1}, g_{\sigma_2}) = 0$ in $TL_n(q)$ we see that e'_n is a homomorphism, hence, epimorphism. After setting N''_n to be the ideal in $\widehat{TL}_{n+1}(q)$, generated by $g_{\sigma_1}g_{\sigma_2}...g_{\sigma_n} - g_{a_{n+1}}g_{\sigma_1}...g_{\sigma_{n-1}}$, we get:

$$
\begin{tikzcd}
\widehat{H}_{n+1}(q) \arrow[d, two heads] & = & H_n(q) \arrow[d, two heads, "\theta_n"'] & \oplus & N'_n \arrow[d, two heads] \\
\widehat{TL}_{n+1}(q) & = & TL_n(q) & \oplus & N''_n
\end{tikzcd}
$$

5.2.3 The tower of affine TL-algebras

We consider the monomorphism $F : \widehat{H}_n(q) \longrightarrow \widehat{H}_{n+1}(q)$ of proposition 4.3.3, which allows us to define the map:

$$
\begin{aligned}
F_n : \widehat{TL}_n(q) &\longrightarrow \widehat{TL}_{n+1}(q) \\
t_{\sigma_i} &\longmapsto g_{\sigma_i} \text{ for } 1 \leq i \leq n-1 \\
t_{a_n} &\longmapsto g_{\sigma_n} g_{a_{n+1}} g_{\sigma_n}^{-1}.
\end{aligned}
$$

Lemma 5.2.1. *F_n is a an algebra homomorphism.*

Proof. In order to show that F_n is an algebra homomorphism, we are reduced to show that:

$$F_n(V(t_{a_n}, t_{\sigma_1})) = 0 \text{ and } F_n(V(t_{a_n}, t_{\sigma_{n-1}})) = 0.$$

Starting by the first term:

$$
\begin{aligned}
&F_n \ (\ t_{\sigma_1} t_{a_n} t_{\sigma_1} + t_{a_n} t_{\sigma_1} + t_{\sigma_1} t_{a_n} + t_{a_n} + t_{\sigma_1} + 1) \\
&= g_{\sigma_1} g_{\sigma_n} g_{a_{n+1}} g_{\sigma_n}^{-1} g_{\sigma_1} + g_{\sigma_1} g_{\sigma_1} g_{a_{n+1}} g_{\sigma_1}^{-1} + g_{\sigma_1} g_{a_{n+1}} g_{\sigma_1}^{-1} g_1 + g_{\sigma_1} g_{a_{n+1}} g_n^{-1} + g_{\sigma_1} + 1 \\
&= \frac{1}{q} g_{\sigma_1} g_{\sigma_n} g_{a_{n+1}} g_{\sigma_n} g_{\sigma_1} + \frac{1}{q} g_{\sigma_1} g_{\sigma_n} g_{a_{n+1}} g_{\sigma_n} + (1/q) g_{\sigma_n} g_{a_{n+1}} g_{\sigma_n} g_{\sigma_1} + \frac{1}{q} g_{\sigma_n} g_{a_{n+1}} g_{\sigma_n} \\
&\quad + g_{\sigma_1} + 1 + \frac{1-q}{q} g_{\sigma_1} g_{\sigma_n} g_{a_{n+1}} g_{\sigma_1} + \frac{1-q}{q} g_{\sigma_n} g_{\sigma_n} g_{a_{n+1}} + \frac{1-q}{q} g_{\sigma_n} g_{a_{n+1}} g_{\sigma_n} + \frac{1-q}{q} g_{\sigma_n} g_{a_{n+1}}.
\end{aligned}
$$

Hence, $F_n\Big(V(t_{a_n}, t_{\sigma_1})\Big)$ is equal to:

$$
\begin{aligned}
&\frac{1}{q} g_{\sigma_n} (g_{\sigma_1} g_{a_{n+1}} g_{\sigma_1} + g_{\sigma_1} g_{a_{n+1}} + g_{a_{n+1}} g_{\sigma_1} + g_{a_{n+1}}) g_{\sigma_n} + g_{\sigma_1} + 1 \\
&+ \frac{1-q}{q} g_{\sigma_n} (g_{\sigma_1} g_{a_{n+1}} g_{\sigma_1} + g_{\sigma_1} g_{a_{n+1}} + g_{a_{n+1}} g_{\sigma_1} + g_{a_{n+1}}).
\end{aligned}
$$

Thus,

$$F_n(V(t_{a_n}, t_{\sigma_1})) = (\frac{q-1}{q} g_{\sigma_n} - \frac{q-1}{q} g_{\sigma_n} + 1 - 1)(1 + g_{\sigma_1}) = 0.$$

For the second relation, we have:

$$F_n \; (\; t_{\sigma_{n-1}} t_{a_n} t_{\sigma_{n-1}} + t_{a_n} t_{\sigma_{n-1}} + t_{\sigma_{n-1}} t_{a_n} + t_{a_n} + t_{\sigma_{n-1}} + 1)$$

$$= g_{\sigma_{n-1}} g_{\sigma_n} g_{a_{n+1}} g_n^{-1} g_{\sigma_{n-1}} + g_{\sigma_{n-1}} g_{\sigma_n} g_{a_{n+1}} g_n^{-1} + g_{\sigma_n} g_{a_{n+1}} g_n^{-1} g_{\sigma_{n-1}} + g_{\sigma_n} g_{a_{n+1}} g_n^{-1} + g_{\sigma_{n-1}} + 1$$

$$= \frac{1}{q} g_{\sigma_{n-1}} \underbrace{g_{\sigma_n} g_{a_{n+1}} g_{\sigma_n}} g_{\sigma_{n-1}} + \frac{1}{q} g_{\sigma_{n-1}} g_{\sigma_n} g_{a_{n+1}} g_{\sigma_n} + \frac{1}{q} g_{\sigma_n} g_{a_{n+1}} g_{\sigma_n} g_{\sigma_{n-1}} + \frac{1}{q} g_{\sigma_n} g_{a_{n+1}} g_{\sigma_n}$$
$$+ g_{\sigma_{n-1}} + 1 + \frac{q-1}{q} g_{\sigma_{n-1}} g_{\sigma_n} g_{a_{n+1}} g_{\sigma_{n-1}} + \frac{q-1}{q} g_{\sigma_{n-1}} g_{\sigma_n} g_{a_{n+1}} + \frac{q-1}{q} g_{\sigma_n} g_{a_{n+1}} g_{\sigma_{n-1}}$$
$$+ \frac{q-1}{q} g_{\sigma_n} g_{a_{n+1}}.$$

The braid relations show us that this has the same shape as the first relation.

$$\text{So, } F_n(V(t_{a_n}, t_{\sigma_{n-1}})) = (\frac{q-1}{q} g_{a_{n+1}} - \frac{q-1}{q} g_{a_{n+1}} + 1 - 1)(1 + g_{\sigma_{n-1}}) = 0.$$

□

Now we can talk of the tower of affine T-L algebras, (we do not know yet, whether it is a tower of faithful arrows or not):

$$\widehat{TL}_1(q) \xrightarrow{F_1} \widehat{TL}_2(q) \xrightarrow{F_2} \widehat{TL}_3(q) \longrightarrow \quad ... \quad \widehat{TL}_n(q) \xrightarrow{F_n} \widehat{TL}_{n+1}(q) \longrightarrow ...$$

Recall the surjection e'_n of $\widehat{TL}_{n+1}(q)$ onto $TL_n(q)$. This surjection respects the homomorphic images of $\widehat{TL}_n(q)$ (resp. $\widehat{TL}_{n+1}(q)$) in $TL_{n-1}(q)$ (resp. $TL_n(q)$), that is: for all $n \geq 1$, the following diagram is commutative:

$$\begin{tikzcd}
\widehat{TL}_n(q) \arrow[r, "F_n"] \arrow[d, two heads] & \widehat{TL}_{n+1}(q) \arrow[d, two heads] \\
TL_{n-1}(q) \arrow[r, hook] & TL_n(q)
\end{tikzcd}$$

More generally, we have the following commutative diagram:

$$
\begin{tikzcd}
& K[B(\widetilde{A_{n-1}})] \arrow[rr, hook] \arrow[dl, two heads] \arrow[dd, two heads] & & K[B(\tilde{A}_n)] \arrow[dl, two heads] \arrow[dd, two heads] \\
K[B(A_{n-1})] \arrow[rr, hook, crossing over] \arrow[dd, two heads] & & K[B(A_{n-1})] & \\
& \widehat{H}_n(q) \arrow[rr, hook] \arrow[dl, two heads] \arrow[dd, two heads] & & \widehat{H}_{n+1}(q) \arrow[dl, two heads] \arrow[dd, two heads] \\
H_{n-1}(q) \arrow[rr, hook, crossing over] \arrow[dd, two heads] & & H_n(q) \arrow[from=uu, two heads, crossing over] & \\
& \widehat{TL}_n(q) \arrow[rr] \arrow[dl, two heads] & & \widehat{TL}_{n+1}(q) \arrow[dl, two heads] \\
TL_{n-1}(q) \arrow[rr] & & TL_n(q) \arrow[from=uu, two heads, crossing over] &
\end{tikzcd}
$$

5.3 Bibliographical remarks and problems

We follow [GL01] when stating the general definition 5.0.1 (introduced by Graham) and proposition 5.0.2. For the definitions and presentations of 5.1 we have followed [GL03] and [GL98].

Actually, we could have started with a T-L algebra following Jones, by choosing a ground ring $K = \mathbb{Q}[\sqrt{t}]$ for some parameter t, and defining $TL_n(q)$ by the generators $\{g_{\sigma_1}, \dots, g_{\sigma_n}\}$ and relations of 5.1 but changing the quadratic relations by:

$g_{\sigma_i}^2 = \sqrt{t}(t-1)g_{\sigma_i} - t^2$, for $1 \leq i \leq n$. Then the Markov conditions resulting from proposition 5.1.3, would have been:

(1) $\tau_1(1) = 1$.

(2) $\tau_{n+1}(hg_{\sigma_n}^{\pm 1}) = \tau_n(h)$, for any h in $TL_{n-1}(q)$.

(3) For $1 \leq n$, the trace $\tau_{n+1} : TL_n(q) \longrightarrow K$ verifies:

$$\tau_{n+1}(bT_{\sigma_n}c) = \tau_n(bc) \text{ and } \tau_{n+1}(a) = -\tfrac{1+t}{\sqrt{t}}\tau_n(a).$$

But anyway, we choose to define the trace on the Hecke algebra, then to see it factorizing through the T-L algebra, and that is why we introduced S and P (or equivalently q' and q'').

Many indications lead to the assumption that, the affine T-L tower is a tower of inclusions, i.e., that : $\widehat{TL}_n(q)$ injects into$\widehat{TL}_{n+1}(q)$, no proof was found, yet.

وَهَمَّ .لوْلا ان رَّاى بُرْهَانَ رَبِّهِ. (يوسف)

Il était sur le point de ...

sauf que ...

il reçut la preuve, des cieux. (Joseph)

Chapter 6

Affine Markov traces over $\tilde{A}$-type Temperley-Lieb algebras

In this chapter we define an affine Markov trace, then we give an example and finally, we prove that this example is the unique affine Markov trace, that is theorem 6.2.8.

6.1 On the space of traces over $\widehat{TL}_{n+1}(q)$

6.1.1 Markov elements

The aim of this subsection is to prove the following theorem.

Theorem 6.1.1. * *Let τ_{n+1} be any trace over $\widehat{TL}_{n+1}(q)$ for $2 \leq n$. Then, τ_{n+1} is uniquely defined by its values over the "Markov elements" in $\widehat{TL}_{n+1}(q)$.*

We consider $F_n : \widehat{TL}_n(q) \longrightarrow \widehat{TL}_{n+1}(q)$ of 4.2.3. In this subsection we set $F := F_n$. We give a definition of Markov elements in $\widehat{TL}_{n+1}(q)$ for $2 \leq n$. Then we show that any trace over $\widehat{TL}_{n+1}(q)$ is uniquely determined by its values over those elements.

Definition 6.1.2. *For F as above, and $2 \leq n$. a Markov element in $\widehat{TL}_{n+1}(q)$ is any element of the form $Ag^{\epsilon}_{\sigma_n}B$. Where A and B are in $F(\widehat{TL}_n(q))$ and $\epsilon \in \{0, 1\}$.*

The proof of theorem 6.1.1 is divided into three parts. In the first we show some general facts, in the second we prove the above theorem for $n = 2$ and in the third part we prove the theorem for $3 \leq n$.

Part 1

In this part, we suppose that τ_{n+1} is any trace over over $\widehat{TL}_{n+1}(q)$. We will apply τ_{n+1} to $\widehat{TL}_{n+1}(q)$ assuming that $2 \leq n$, and show that τ_{n+1} is uniquely determined on $\widehat{TL}_{n+1}(q)$ by its values on the positive powers of $g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}$, in addition to its values on Markov elements. From now on we denote by w: an arbitrary element in $W^c(\tilde{A}_n)$.

Lemma 6.1.3. *In $\widehat{TL}_{n+1}(q)$ we have:*

$$(1)\ g_{\sigma_n}(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^k = (q-1)(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^k + \sum_{i=1}^{i=k-1} f_i (g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^i + A\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1} F(t_{a_n})\Big)^k g_{\sigma_n} \prod_{j=0}^{j=k-1} \psi^j\Big[F((t_{a_n})^{-1})\Big].$$

$$(2)\ (g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^k g_{\sigma_n} = (q-1)(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^k + \sum_{i=1}^{i=k-1} h_i (g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^i + A\prod_{j=0}^{j=k-1} \phi^j\Big[(\sigma_{n-1})^{-1}\Big] g_{\sigma_n}\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1} F(t_{a_n})\Big)^k.$$

With A in the ground field, f_i, h_i in $F(\widehat{TL}_n(q))$ and $\phi^{-1} = \psi$.

Proof.

$$\begin{aligned} g_{\sigma_n}\Big(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}\Big)^k &= \Big(q-1\Big)\Big(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}\Big)^k \\ &\quad + q g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1} F\Big(t_{a_n}\Big) g_{\sigma_n} F\Big((t_{a_n})^{-1}\Big)\Big(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}\Big)^{k-1} \\ &= \Big(q-1\Big)\Big(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}\Big)^k + \\ &\quad q g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1} F\Big(t_{a_n}\Big) g_{\sigma_n}\Big(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}\Big)^{k-1} \psi^{k-1}\Big[F((t_{a_n})^{-1})\Big]. \end{aligned}$$

So, by induction on k, (1) follows, in the very same way we deal with (2), by noticing that: $g_{a_{n+1}} g_{\sigma_n} = g_{\sigma_n}^{-1} F(t_{a_n}) g_{\sigma_n}^2 = (q-1) g_{a_{n+1}} + q g_{\sigma_n}^{-1} F(t_{a_n})$.

□

Recall corollary 3.5.2, we deduce that any element of the basis $\widehat{TL}_{n+1}(q)$, which is not a positive power of $g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}$, is either of the form $c(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^k g_{\sigma_n\sigma_{n-1}..\sigma_i}$, or of the form $g_{\sigma_{i_0}..\sigma_2\sigma_1 a_{n+1}}(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^k d g_{\sigma_n\sigma_{n-1}..\sigma_i}$, where c and d are in $F(\widehat{TL}_n(q))$, $1 \le i \le n+1$ and $0 \le i_0 \le n-1$.

By lemma 6.1.3 $c(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^k g_{\sigma_n\sigma_{n-1}..\sigma_i}$ is of the form

$$\sum_{j=1}^{j=h} c_j (g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^j + M.$$

Where $h \le k$, c_j is in $F(\widehat{TL}_n(q))$ for any j and M is a Markov element.

Now we deal with the second form:

$$\tau_{n+1}\Big(g_{\sigma_{i_0}..\sigma_2\sigma_1 a_{n+1}} c(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^k g_{\sigma_n\sigma_{n-1}..\sigma_i}\Big) = \tau_{n+1}\Big(g_{\sigma_n\sigma_{n-1}..\sigma_i} g_{\sigma_{i_0}..\sigma_2\sigma_1 a_{n+1}} c(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^k\Big).$$

For any possible value for i_0 or i, we see that:

$$g_{\sigma_n\sigma_{n-1}..\sigma_i} g_{\sigma_{i_0}..\sigma_2\sigma_1 a_{n+1}} c(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^k = c' g_{\sigma_n} (g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^s c''.$$

Where c', c'' are in $F(\widehat{TL}_n(q))$ and $s \leq k+1$. By lemme 6.1.3 we see that this element is of the form:

$$\sum_{j=1}^{j=h} f_j (g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^j + M,$$

where $h \leq k+1$, f_j is in $F(\widehat{TL}_n(q))$ for any j and M is a Markov element .

Since we are dealing with elements which are not Markov elements, we see that, in order to define τ_{n+1} uniquely, we are reduced to compute $\tau_{n+1}\Big(\Omega(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^k\Big)$, where $1 \leq k$ (since if k is equal to 0 then we are again in the case of a Markov element) and Ω is in $F\Big(\widehat{TL}_n(q)\Big)$.

Lemma 6.1.4. *Let $2 \leq n$ then τ_{n+1} is uniquely defined by its values on Markov elements, in addition to its values on $(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^k$, with $0 \leq k$.*

Proof. In order to determine $= \tau_{n+1}\Big(h(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^k\Big)$, with a positive k and an arbitrary h in $F\Big(\widehat{TL}_n(q)\Big)$, it is enough to treat $\tau_{n+1}\Big(F(t_x)(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^k\Big)$, with x in $W^c(\tilde{A}_{n-1})$, but the fact that τ_{n+1} is a trace, in addition to the fact that $g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}$ acts as a Dynkin automorphism on $F\Big(\widehat{TL}_n(q)\Big)$, authorizes us to suppose that x has a reduced expression which ends with σ_{n-1}.

Now we show by induction on $l(x)$, that $\tau_{n+1}\Big(F(t_x)(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^k\Big)$ is a sum of values of τ_{n+1} over $(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^k$, elements of the form $h(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^i$ with $i < k$ and Markov elements, (of course with coefficients in the ground field which might be zeros).

For $l(x) = 0$ the hypothesis is valid. Take $l(x) > 0$, and let $x = z\sigma_{n-1}$ be a reduced expression, hence:

$$\begin{aligned}\tau_{n+1}\Big(F(t_x)(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^k\Big) &= \tau_{n+1}\Big(F(t_z)F(t_{\sigma_{n-1}})g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^{k-1}\Big)\\ &= \tau_{n+1}\Big(F(t_z)\underbrace{g_{\sigma_{n-1}}g_{\sigma_n}g_{\sigma_{n-1}}}_{=-V(g_{\sigma_{n-1}},g_{\sigma_n})} g_{\sigma_{n-2}..\sigma_1 a_{n+1}}(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^{k-1}\Big).\end{aligned}$$

This is equal to the following sum:

$$
\begin{aligned}
&-\tau_{n+1}\Big(F(t_z)(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^k\Big)\\
&-\tau_{n+1}\Big(F(t_z)g_{\sigma_{n-1}}g_{\sigma_{n-2}..\sigma_1}g_{a_{n+1}}(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^{k-1}\Big)\\
&-\tau_{n+1}\Big(F(t_z)g_{\sigma_{n-2}..\sigma_1 a_{n+1}}(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^{k-1}\Big)\\
&-\tau_{n+1}\Big(F(t_z)g_{\sigma_{n-1}}g_{\sigma_{n-2}..\sigma_1}\underbrace{g_{\sigma_n}g_{a_{n+1}}}(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^{k-1}\Big)\\
&-\tau_{n+1}\Big(F(t_z)g_{\sigma_{n-2}..\sigma_1}\underbrace{g_{\sigma_n}g_{a_{n+1}}}(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^{k-1}\Big).
\end{aligned}
$$

Now we apply the induction hypothesis to the first term. The second and the third terms are equal to:

$$
\begin{aligned}
&\tau_{n+1}\left(F\Big(t_z\Big)g_{\sigma_{n-1}}g_{\sigma_{n-2}..\sigma_1}F\Big(t_{a_n}\Big)g_{\sigma_n}F\Big((t_{a_n})^{-1}\Big)\Big(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}\Big)^{k-1}\right)\\
&\quad+\tau_{n+1}\left(F\Big(t_z\Big)g_{\sigma_{n-2}..\sigma_1}F\Big(t_{a_n}\Big)g_{\sigma_n}F\Big((t_{a_n})^{-1}\Big)\Big(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}\Big)^{k-1}\right),
\end{aligned}
$$

which is equal to:

$$
\begin{aligned}
&\tau_{n+1}\left(\psi^{1-k}\Big[F\Big((t_{a_n})^{-1}\Big)\Big]F\Big(t_z\Big)g_{\sigma_{n-1}}g_{\sigma_{n-2}..\sigma_1}F\Big(t_{a_n}\Big)\Big(g_{\sigma_n}(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^{k-1}\Big)\right)\\
&\quad+\tau_{n+1}\left(\psi^{1-k}\Big[F\Big((t_{a_n})^{-1}\Big)\Big]F\Big(t_z\Big)g_{\sigma_{n-2}..\sigma_1}F(t_{a_n})\Big(g_{\sigma_n}(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^{k-1}\Big)\right).
\end{aligned}
$$

The fourth and the fifth terms are equal to:

$$
\begin{aligned}
&\tau_{n+1}\left(F\Big(t_z\Big)g_{\sigma_{n-1}}g_{\sigma_{n-2}..\sigma_1}F\Big(t_{a_n}\Big)\Big(g_{\sigma_n}(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^{k-1}\Big)\right)\\
&\quad+\tau_{n+1}\left(F\Big(t_z\Big)g_{\sigma_{n-2}..\sigma_1}F\Big(t_{a_n}\Big)\Big(g_{\sigma_n}(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^{k-1}\Big)\right).
\end{aligned}
$$

Thus, lemma 6.1.3 tells us that the hypothesis is valid for those four terms. This step is to be applied repeatedly, to the powers of $g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}$ down to an element of the form $\tau_{n+1}\Big(h(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^1\Big)$, arriving to the sum of:

$$\tau_{n+1}(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}) \text{ and } \tau_{n+1}(h' g_{\sigma_{n-1}..\sigma_1 a_{n+1}}),$$

which is the sum of values of τ_{n+1} on Markov elements, since $h, h' \in F\Big(\widehat{TL}_n(q)\Big)$. □

We end this part by the following lemma:

Lemma 6.1.5. *Let $1 \leq k$. Then $(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^k$ is a sum of two kinds of elements:*

(1) $g_{\sigma_n}\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n})\Big)^j g_{\sigma_n}h$, *with $j \leq k$.*

(2) $\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n})\Big)^i g_{\sigma_n}f$, *with $i < k$,*

with h, f in $F\Big(\widehat{TL}_n(q)\Big)$ and $2 \leq n$.

Moreover, in the first type we have one, and only one element, in which $j = k$, in which we have:

$$h = \prod_{i=0}^{i=k-1} \phi^i\Big[F(t_{a_n}^{-1})\Big].$$

Proof. Suppose that $k = 1$. Then,

$$g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}} = g_{\sigma_n}\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n})\Big)\sigma_n F\Big(t_{a_n}\Big)^{-1},$$

The hypothesis is valid.

Suppose the hypothesis is valid for $k-1$, then, with $2 \leq k$, we have:

$$(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^k = (g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^{k-1} g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}},$$

here, when applying the hypothesis to $(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^{k-1}$ we have two cases:

(1) $g_{\sigma_n}\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n})\Big)^{j'} g_{\sigma_n}h g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}$, with $j' \leq k-1$ which is: $g_{\sigma_n}\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n})\Big)^{j'} g_{\sigma_n}g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}\psi^{-1}\Big[h\Big]$, which is equal to:

$$q g_{\sigma_n}\ \Big(\ g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n})\Big)^{j'+1} g_{\sigma_n}F\Big((t_{a_n})^{-1}\Big)\psi^{-1}\Big[h\Big]$$

$$+\,(q-1)g_{\sigma_n}g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}\psi^{-1}\left[\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n})\Big)^{j'}\right]\psi^{-1}\left[h\right].$$

Since, $j'+1 \leq k$, the first term is clear to be of the first type, while the second term is equal to:

$$\Big(q-1\Big)q g_{\sigma_{n-1}..\sigma_1}F\Big(t_{a_n}\Big)g_{\sigma_n}F\Big((t_{a_n})^{-1}\Big)\psi^{-1}\Big[\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n})\Big)^{j'}\Big]\psi^{-1}\Big[h\Big]+$$

$$\Big(q-1\Big)^2 g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}\psi^{-1}\Big[\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n})\Big)^{j'}\Big]\psi^{-1}\Big[h\Big].$$

Here, the first term is of the second type (with $i = 1 < k$), and the second term is of the first type (with $j = 1$).

(2) $\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n})\Big)^{i'} g_{\sigma_n} f g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}$, with $i' < k-1$, which is:

$$\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n})\Big)^{i'} g_{\sigma_n} g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}} \psi^{-1}\Big[f\Big] =$$

$$q\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n})\Big)^{i'+1} g_{\sigma_n} F\Big((t_{a_n})^{-1}\Big)\psi^{-1}\Big[f\Big]+$$

$$\Big(q-1\Big)g_{\sigma_n}\Big(g_{\sigma_{n-1}..\sigma_1}F(t_{a_n})\Big)g_{\sigma_n}F\Big((t_{a_n})^{-1}\Big)\psi^{-1}\Big[\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n})\Big)^{i'}\Big]\psi^{-1}\Big[f\Big].$$

Since $i'+1 < k$, the first term is of the second type, while the second term is of the first type $j = 1$. The lemma is proven to be true.

(By induction over k again, the proof of the last formula is obvious).

□

Part 2

In this part we will consider a given trace τ_3 over $\widehat{TL}_3(q)$. The aim is to show that τ_3 is uniquely defined by its values on Markov elements. consider

$$\begin{aligned} F_2 : \widehat{TL}_2(q) &\longrightarrow \widehat{TL}_3(q) \\ t_{\sigma_1} &\longmapsto g_{\sigma_1} \\ t_{a_n} &\longmapsto g_{\sigma_2} g_{a_3} g_{\sigma_2}^{-1}. \end{aligned}$$

In this part we will denote to F_2 by F.

Lemma 6.1.4 tells that we can uniquely determine τ_3 by its values over $(g_{\sigma_2\sigma_1 a_3})^k$ for a positive k beside its values on Markov elements. We know as well by lemma 6.1.5 that $(g_{\sigma_2\sigma_1 a_3})^k$ is a sum of two kinds of elements:

(1) $g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^j g_{\sigma_2} h$ with $j \leq k$.

(2) $\Big(g_{\sigma_1}F(t_{a_2})\Big)^i g_{\sigma_2} f$ with $i < k$.

Here, h and f are in $F\Big(\widehat{TL}_2(q)\Big)$.

Moreover, in first type, only when $j = k$, we have:

$$h = \prod_{i=0}^{i=k-1} \psi^i \left[\left(F(t_{a_2})^{-1} \right) \right].$$

In other terms:

$$\begin{aligned} g_{\sigma_2}\Big(g_{\sigma_1}F\big(t_{a_2}\big)\Big)^k g_{\sigma_2} = {} & \Big(g_{\sigma_2\sigma_1 a_3}\Big)^k \prod_{i=0}^{i=k-1} \psi^i \left[\Big(F(t_{a_2}) \Big) \right] \\ & - \sum_{r=1}^{r=k-1} \Big(g_{\sigma_1}F\big(t_{a_2}\big)\Big)^r g_{\sigma_2} f_r \prod_{i=0}^{i=k-1} \psi^i \left[\Big(F(t_{a_2}) \Big) \right] \\ & + \sum_{l=1}^{l=k-1} g_{\sigma_2}\Big(g_{\sigma_1}F\big(t_{a_2}\big)\Big)^l g_{\sigma_2} f'_l \prod_{i=0}^{i=k-1} \psi^i \left[\Big(F(t_{a_2}) \Big) \right]. \end{aligned}$$

We repeat the same step on $g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^l g_{\sigma_2}$ for every l. We deduce the following:

Corollary 6.1.6. *For every $h > 0$, we have: $g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^h g_{\sigma_2} = \sum_{j=0}^{j=h} c_j \Big(g_{\sigma_2\sigma_1 a_3}\Big)^j + \sum_i M_i$.*

Here, c_j is in $F\Big(\widehat{TL}_2(q)\Big)$ for every j, and M_i is a Markov element for every i.

Our way to prove Theorem 6.1.1 for $n = 3$, is to show that $\tau_3\Big((g_{\sigma_2\sigma_1 a_3})^k\Big)$ is a linear combination of values of τ_3 on Markov elements and values on elements of the form $c(g_{\sigma_2\sigma_1 a_3})^h$, where $h < k$ and c in $F\Big(\widehat{TL}_2(q)\Big)$. Then, using the induction in the proof of Lemma 6.1.4, beside the fact that $\tau_3(g_{\sigma_2\sigma_1 a_3})$ is a linear combination of some values of τ_3 on Markov elements, we see that the work is done.

Lemma 6.1.7. *Suppose that r and s are positives ,such that $r \leq s$. Then:*

$$\tau_3\Big(\Big(g_{\sigma_1}F(t_{a_2})\Big)^r g_{\sigma_1} g_{\sigma_2} \Big(g_{\sigma_1}F(t_{a_2})\Big)^s g_{\sigma_1} g_{\sigma_2}\Big) = \sum_{j=0}^{j=h} c_j \Big(g_{\sigma_2\sigma_1 a_3}\Big)^j + \sum_i M_i,$$

where $h \leq s$, c_j is in $F\Big(\widehat{TL}_2(q)\Big)$ for every j and M_i is a Markov element for every i.

Proof.

$$\tau_3\Big(\big(g_{\sigma_1}\ F\ (t_{a_2})\big)^r g_{\sigma_1}g_{\sigma_2}\big(g_{\sigma_1}F(t_{a_2})\big)^s g_{\sigma_1}g_{\sigma_2}\Big) =$$

$$= \tau_3\Big(\big(g_{\sigma_1}F(t_{a_2})\big)^r g_{\sigma_1}g_{\sigma_2}\big(g_{\sigma_1}F(t_{a_2})\big)^s g_{\sigma_1}\ \underbrace{g_{\sigma_2\sigma_1a_3}\big(g_{\sigma_2\sigma_1a_3}\big)^{-1}}\ g_{\sigma_2}\Big)$$

$$= \tau_3\Big(\big(g_{\sigma_1}F(t_{a_2})\big)^r g_{\sigma_1}g_{\sigma_2}g_{\sigma_2\sigma_1a_3}\psi\Big[\big(g_{\sigma_1}F(t_{a_2})\big)^s g_{\sigma_1}\Big]\big(g_{\sigma_2\sigma_1a_3}\big)^{-1} g_{\sigma_2}\Big)$$

$$= \tau_3\Big(\big(g_{\sigma_1}F(t_{a_2})\big)^r g_{\sigma_1}g_{\sigma_2}g_{\sigma_2\sigma_1a_3}\psi\Big[\big(g_{\sigma_1}F(t_{a_2})\big)^s g_{\sigma_1}\Big]g_{a_3}^{-1}g_{\sigma_1}^{-1}\Big)$$

$$= \tau_3\Big(\big(g_{\sigma_1}F(t_{a_2})\big)^{r-1} g_{\sigma_1}g_{\sigma_2}^2g_{\sigma_1}g_{a_3}\big(F(t_{a_2})g_{\sigma_1}\big)^s F\big(t_{a_2}\big)g_{a_3}^{-1}F\big(t_{a_2}\big)\Big)$$

$$= \frac{1-q}{q}\tau_3\Big(\big(g_{\sigma_1}F(t_{a_2})\big)^{r-1} g_{\sigma_1}g_{\sigma_2}^2g_{\sigma_1}g_{a_3}\big(F(t_{a_2})g_{\sigma_1}\big)^s F\big(t_{a_2}\big)F\big(t_{a_2}\big)\Big)$$

$$+ \frac{1}{q}\tau_3\Big(\big(g_{\sigma_1}F(t_{a_2})\big)^{r-1} g_{\sigma_1}g_{\sigma_2}^2g_{\sigma_1}g_{a_3}\big(F(t_{a_2})g_{\sigma_1}\big)^s F\big(t_{a_2}\big)g_{a_3}F\big(t_{a_2}\big)\Big).$$

Now, the term corresponding to $\frac{1-q}{q}$ is τ_3 evaluated on the sum of Markov element and an element of style $c_j(g_{\sigma_2\sigma_1a_3})^1$. So, We are reduced to the second term, thus, reduced to:

$$\tau_3\Big(\big(g_{\sigma_1}F(t_{a_2})\big)^{r-1} g_{\sigma_1}g_{\sigma_2}^2g_{\sigma_1}g_{a_3}\big(F(t_{a_2})g_{\sigma_1}\big)^s F\big(t_{a_2}\big)g_{a_3}F\big(t_{a_2}\big)\Big).$$

Obviously, we are in the case:

$$q\tau_3\Big(\big(g_{\sigma_1}F(t_{a_2})\big)^{r-1} g_{\sigma_1}g_{\sigma_1}g_{a_3}\big(F(t_{a_2})g_{\sigma_1}\big)^s F\big(t_{a_2}\big)g_{a_3}F\big(t_{a_2}\big)\Big) =$$

$$q\tau_3\Big(\big(g_{\sigma_1}F(t_{a_2})\big)^{r-1} g_{\sigma_1}^2g_{a_3}\big(F(t_{a_2})g_{\sigma_1}\big)^s F\big(t_{a_2}^2\big)g_{\sigma_2}\Big),$$

since $g_{a_3}F\big(t_{a_2}\big) = F\big(t_{a_2}\big)g_{\sigma_2}$.

Now,

$$\tau_3\left(\left(g_{\sigma_1}F(t_{a_2})\right)^{r-1}g_{\sigma_1}^2 g_{a_3}\left(F(t_{a_2})g_{\sigma_1}\right)^s F\left(t_{a_2}^2\right)g_{\sigma_2}\right) =$$

$$\left(q-1\right)\tau_3\left(\left(g_{\sigma_1}F(t_{a_2})\right)^{r-1}g_{\sigma_1}^2 g_{a_3}F\left(t_{a_2}\right)\left(g_{\sigma_1}F(t_{a_2})\right)^s g_{\sigma_2}\right)$$

$$+\ q\tau_3\left(\left(g_{\sigma_1}F(t_{a_2})\right)^{r-1}g_{\sigma_1}^2 g_{a_3}F(t_{a_2})g_{\sigma_1}\left(F(t_{a_2})g_{\sigma_1}\right)^{s-1}g_{\sigma_2}\right).$$

Which is equal to the sum:

$$\left(q-1\right)\tau_3\left(\left(g_{\sigma_1}F(t_{a_2})\right)^{r-1}g_{\sigma_1}^2 F\left(t_{a_2}\right)\underbrace{g_{\sigma_2}\left(g_{\sigma_1}F(t_{a_2})\right)^s g_{\sigma_2}}\right)$$

$$+\,q\tau_3\left(\left(g_{\sigma_1}F(t_{a_2})\right)^{r-1}g_{\sigma_1}^2 F\left(t_{a_2}\right)g_{\sigma_2}g_{\sigma_1}\left(F(t_{a_2})g_{\sigma_1}\right)^{s-1}g_{\sigma_2}\right).$$

Now, the first term is covered by corollary 6.1.6. Thus we are interested with the second term:

$$\tau_3\left(\left(g_{\sigma_1}F(t_{a_2})\right)^{r-1}g_{\sigma_1}^2\ F\left(t_{a_2}\right)g_{\sigma_2}\left(g_{\sigma_1}F(t_{a_2})\right)^{s-1}g_{\sigma_1}g_{\sigma_2}\right) =$$

$$q\tau_3\left(\left(g_{\sigma_1}F(t_{a_2})\right)^{r}g_{\sigma_2}\left(g_{\sigma_1}F(t_{a_2})\right)^{s-1}g_{\sigma_1}g_{\sigma_2}\right)$$

$$+\left(q-1\right)\tau_3\left(\left(g_{\sigma_1}F(t_{a_2})\right)^{r-2}g_{\sigma_1}F(t_{a_2}^2)g_{\sigma_2}\left(g_{\sigma_1}F(t_{a_2})\right)^{s-1}g_{\sigma_1}g_{\sigma_2}\right).$$

Which is equal to:

$$q\tau_3\left(\underbrace{g_{\sigma_2}\left(g_{\sigma_1}F(t_{a_2})\right)^{r}g_{\sigma_2}}\left(g_{\sigma_1}F(t_{a_2})\right)^{s-1}g_{\sigma_1}\right)+$$

$$\left(q-1\right)\tau_3\left(\left(g_{\sigma_1}F(t_{a_2})\right)^{r-2}g_{\sigma_1}F\left(t_{a_2}^2\right)g_{\sigma_2}\left(g_{\sigma_1}F(t_{a_2})\right)^{s-1}g_{\sigma_1}g_{\sigma_2}\right)$$

The first term is covered by corollary 6.1.6. We are reduced to

$$\left(\tau_3\left(g_{\sigma_1}F(t_{a_2})\right)^{r-2}g_{\sigma_1}F\left(t_{a_2}^2\right)g_{\sigma_2}\left(g_{\sigma_1}F(t_{a_2})\right)^{s-1}g_{\sigma_1}g_{\sigma_2}\right),$$

which is equal to:

$$\Big(q-1\Big)\tau_3\Bigg(\underbrace{g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^{r-1}g_{\sigma_2}}\Big(g_{\sigma_1}F(t_{a_2})\Big)^{s-1}g_{\sigma_1}\Bigg)+$$

$$q\tau_3\Bigg(\Big(g_{\sigma_1}F(t_{a_2})\Big)^{r-2}g_{\sigma_1}g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^{s-1}g_{\sigma_1}g_{\sigma_2}\Bigg).$$

The first term is covered by corollary 6.1.6. Thus, we see that, in general, the value of τ_3 over $\Big(g_{\sigma_1}F(t_{a_2})\Big)^r g_{\sigma_1}g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^s g_{\sigma_1}g_{\sigma_2}$ can be shifted to its value over:

$$\Big(g_{\sigma_1}F(t_{a_2})\Big)^{r-2}g_{\sigma_1}g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^{s-1}g_{\sigma_1}g_{\sigma_2}.$$

After a finite number of repetitions of the computation above (with the possibility of exchanging r and s), we see that the lemma is proven modulo determining:

$$\tau_3\Bigg(\Big(g_{\sigma_1}F(t_{a_2})\Big)^m \underbrace{g_{\sigma_1}g_{\sigma_2}g_{\sigma_1}}F\Big(t_{a_2}\Big)g_{\sigma_1}g_{\sigma_2}\Bigg).$$

We see that the terms corresponding to $-g_{\sigma_1}$ and -1 correspond to Markov elements. While those who correspond to $-g_{\sigma_1}g_{\sigma_2}$ and $-g_{\sigma_2}$ are covered by corollary 6.1.6 for $h=1$. Finally the term corresponding to $-g_{\sigma_2}g_{\sigma_1}$ is covered by corollary 6.1.6 for $h=m$.

□

Lemma 6.1.8. *Suppose that r and s are positive such that $r \leq s$. Then:*

$$\tau_3\Bigg(g_{a_3}F\Big(t_{a_2}\Big)\Big(g_{\sigma_1}F(t_{a_2})\Big)^s g_{a_3}\Big(g_{\sigma_1}F(t_{a_2})\Big)^r\Bigg)=\sum_{j=0}^{j=h}c_j\Big(g_{\sigma_2\sigma_1 a_3}\Big)^j+\sum_i M_i.$$

Where $h \leq s$, c_j is in $F(\widehat{TL}_2(q))$ for every j and M_i is a Markov element for every i.

Proof.

$$\tau_3\Bigg(g_{a_3}F\Big(t_{a_2}\Big)\Big(g_{\sigma_1}F(t_{a_2})\Big)^s g_{a_3}\Big(g_{\sigma_1}F(t_{a_2})\Big)^r\Bigg)=$$

$$\tau_3\Bigg(g_{a_3}\Big(g_{\sigma_2\sigma_1 a_3}\Big)^{-1}g_{\sigma_2\sigma_1 a_3}F\Big(t_{a_2}\Big)\Big(g_{\sigma_1}F(t_{a_2})\Big)^s g_{a_3}\Big(g_{\sigma_1}F(t_{a_2})\Big)^r\Bigg)=$$

$$\tau_3\Bigg(g_{a_3}\Big(g_{\sigma_2\sigma_1 a_3}\Big)^{-1}\psi\Big[F\Big(t_{a_2}\Big)\Big(g_{\sigma_1}F(t_{a_2})\Big)^s\Big]g_{\sigma_2\sigma_1 a_3}g_{a_3}\Big(g_{\sigma_1}F(t_{a_2})\Big)^r\Bigg)=$$

$$\tau_3\Bigg(g_{\sigma_1}^{-1}g_{\sigma_2}^{-1}g_{\sigma_1}\Big(F(t_{a_2})g_{\sigma_1}\Big)^s g_{\sigma_2}g_{\sigma_1}g_{a_3}^2\Big(g_{\sigma_1}F(t_{a_2})\Big)^r\Bigg).$$

Here, we see that this term is a sum of two terms coming from $g_{a_3}^2 = (q-1)g_{a_3} + q$. The term corresponding to $(q-1)g_{a_3}$ is covered the same way as in the last lemma (with a_3 instead of σ_2 above. Hence we treat the term corresponding to q, that is:

$$\tau_3\left(g_{\sigma_1}^{-1}\,\underbrace{g_{\sigma_2}^{-1}}\,\left(g_{\sigma_1}F(t_{a_2})\right)^s\,\underbrace{g_{\sigma_1}g_{\sigma_2}g_{\sigma_1}}\,\left(g_{\sigma_1}F(t_{a_2})\right)^r\right).$$

Before applying TL relations, we see in the same way as above, that we are reduced to the next value (otherwise, it is τ_3 evaluated on a Markov element):

$$\tau_3\left(g_{\sigma_1}^{-1}\,\underbrace{g_{\sigma_2}}\,\left(g_{\sigma_1}F(t_{a_2})\right)^s\,\underbrace{g_{\sigma_1}g_{\sigma_2}g_{\sigma_1}}\,\left(g_{\sigma_1}F(t_{a_2})\right)^r\right).$$

We see that the terms corresponding to $-g_{\sigma_1}$ and -1 correspond to Markov elements. And those who correspond to $-g_{\sigma_2 g_{\sigma_1}}$ and $-g_{\sigma_2}$, are covered by corollary 6.1.6 for $h = s$.

The term corresponding to $-g_{\sigma_1}g_{\sigma_2}$ is:

$$\tau_3\left(g_{\sigma_1}^{-1}g_{\sigma_2}\left(g_{\sigma_1}F(t_{a_2})\right)^s g_{\sigma_1}g_{\sigma_2}\left(g_{\sigma_1}F(t_{a_2})\right)^r\right),$$

which is:

$$\frac{1-q}{q}\tau_3\left(\left(g_{\sigma_1}F(t_{a_2})\right)^s g_{\sigma_1}\,\underbrace{g_{\sigma_2}\left(g_{\sigma_1}F(t_{a_2})\right)^r g_{\sigma_2}}\right) + \frac{1}{q}\tau_3\left(g_{\sigma_1}g_{\sigma_2}\left(g_{\sigma_1}F(t_{a_2})\right)^s g_{\sigma_1}g_{\sigma_2}\left(g_{\sigma_1}F(t_{a_2})\right)^r\right)$$

The first term is covered by corollary 6.1.6 for $h = r$. The second follows by lemma 6.1.7.

□

Let us go back to $\tau_3(g_{\sigma_2\sigma_1 a_{n+1}})^k$. The aim is to show that:

$$\tau_3\left(g_{\sigma_2\sigma_1 a_{n+1}}\right)^k = \tau_3\left(\sum_{j=0}^{j=h} c_j(g_{\sigma_2\sigma_1 a_3})^j + \Sigma_i M_i\right),$$

where $h < k$. By lemma 6.1.5, it is sufficient to deal with:

$$g_{\sigma_2}\left(g_{\sigma_1}F(t_{a_2})\right)^k g_{\sigma_2}\prod_{i=0}^{i=k-1}\psi^i\left[\left(F(t_{a_2})^{-1}\right)\right].$$

It is clear that this element is written as a linear combination of four kind of elements:

1) $g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^k g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^h$.

2) $g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^k g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^h g_{\sigma_1}$.

3) $g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^k g_{\sigma_2}\Big(F(t_{a_2})g_{\sigma_1}\Big)^h F\Big(t_{a_2}\Big)$.

4) $g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^k g_{\sigma_2}\Big(F(t_{a_2})g_{\sigma_1}\Big)^h$,

where $h \leq [\frac{k}{2}] < k$, since $1 < k$.

1) We start by $\tau_3\Bigg(g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^k g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^h\Bigg)$. Which is equal to:

$$\tau_3\Bigg(\Big(g_{\sigma_1}F(t_{a_2})\Big)^k \underbrace{g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^h g_{\sigma_2}}\Bigg),$$

follows directly, regarding corollary 6.1.6 .

2) Now we consider

$$\tau_3\Bigg(g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^k g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^h g_{\sigma_1}\Bigg), \tag{6.1}$$

which is equal to:

$$\tau_3\Bigg(g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^k g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^h g_{\sigma_1}g_{\sigma_2\sigma_1a_{n+1}}\Big(g_{\sigma_2\sigma_1a_3}\Big)^{-1}\Bigg) =$$

$$\tau_3\Bigg(g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^k g_{\sigma_2}^2 g_{\sigma_1}g_{a_3}\psi\Big[\Big(g_{\sigma_1}F(t_{a_2})\Big)^h g_{\sigma_1}\Big]\Big(g_{\sigma_2\sigma_1a_3}\Big)^{-1}\Bigg) =$$

$$\tau_3\Bigg(\Big(g_{\sigma_1}F(t_{a_2})\Big)^{k-1} g_{\sigma_2}^2 g_{\sigma_1}g_{a_3}\Big(F(t_{a_2})g_{\sigma_1}\Big)^h F(t_{a_2})g_{a_3}^{-1}F\Big(t_{a_2}\Big)\Bigg),$$

with the very same steps as used above, we see that we are reduced to :

$$\tau_3\Bigg(\Big(g_{\sigma_1}F(t_{a_2})\Big)^{k-1} g_{\sigma_1}g_{a_3}\Big(F(t_{a_2})g_{\sigma_1}\Big)^h F\Big(t_{a_2}\Big)g_{a_3}F\Big(t_{a_2}\Big)\Bigg), \text{ which is:}$$

$$\tau_3\Bigg(\Big(g_{\sigma_1}F(t_{a_2})\Big)^{k-1} g_{\sigma_1}\underbrace{g_{a_3}F\Big(t_{a_2}\Big)}\Big(g_{\sigma_1}F(t_{a_2})\Big)^h \underbrace{g_{a_3}F\Big(t_{a_2}\Big)}\Bigg) =$$

$$\tau_3\Big(\big(g_{\sigma_1}F(t_{a_2})\big)^{k-1}g_{\sigma_1}\underbrace{F\big(t_{a_2}\big)g_{\sigma_2}}\big(g_{\sigma_1}F(t_{a_2})\big)^{h}\underbrace{F\big(t_{a_2}\big)g_{\sigma_2}}\Big),\text{ which is equal to:}$$

$$(q-1)\tau_3\Big(\big(g_{\sigma_1}F(t_{a_2})\big)^{k}g_{\sigma_2}\big(g_{\sigma_1}F(t_{a_2})\big)^{h}g_{\sigma_2}\Big)+q\tau_3\Big(\big(g_{\sigma_1}F(t_{a_2})\big)^{k}g_{\sigma_2}\big(g_{\sigma_1}F(t_{a_2})\big)^{h-1}g_{\sigma_1}g_{\sigma_2}\Big),$$

we see that corollary 6.1.6 covers the first term. Thus we see that:

$$\tau_3\Big(g_{\sigma_2}\big(g_{\sigma_1}F(t_{a_2})\big)^{k}g_{\sigma_2}\big(g_{\sigma_1}F(t_{a_2})\big)^{h}g_{\sigma_1}\Big)\text{ in (eq. 6.1), is shifted to:}$$

$$\tau_3\Big(g_{\sigma_2}\big(g_{\sigma_1}F(t_{a_2})\big)^{k}g_{\sigma_2}\big(g_{\sigma_1}F(t_{a_2})\big)^{h-1}g_{\sigma_1}\Big),$$

going on in this manner, we arrive to:

$$\tau_3\Big(g_{\sigma_2}\big(g_{\sigma_1}F(t_{a_2})\big)^{k}g_{\sigma_2}g_{\sigma_1}F\big(t_{a_2}\big)g_{\sigma_1}\Big),$$

with the same steps above, we see that we are reduced to

$$\tau_3\Big(\big(g_{\sigma_1}F\big(t_{a_2}\big)\big)^{k}g_{\sigma_2}g_{\sigma_1}F(t_{a_2}^2)g_{\sigma_2}\Big),\text{ which is equal to}$$

$$\big(q-1\big)\tau_3\Big(\big(g_{\sigma_1}F(t_{a_2})\big)^{k}g_{\sigma_2}g_{\sigma_1}F\big(t_{a_2}\big)g_{\sigma_2}\Big)+q\tau_3\Big(\big(g_{\sigma_1}F(t_{a_2})\big)^{k}g_{\sigma_2}g_{\sigma_1}g_{\sigma_2}\Big),$$

corollary 6.1.6 and TL relations end the job.

3) Here we deal with $\tau_3\Big(g_{\sigma_2}\big(g_{\sigma_1}F(t_{a_2})\big)^{k}g_{\sigma_2}\big(F(t_{a_2})g_{\sigma_1}\big)^{h}F\big(t_{a_2}\big)\Big)$, which is:

$$\begin{aligned}&\tau_3\Big(F\big(t_{a_2}\ \big)\ g_{\sigma_2}\big(g_{\sigma_1}F(t_{a_2})\big)^{k}g_{\sigma_2}\big(F(t_{a_2})g_{\sigma_1}\big)^{h}\Big)\\ &\qquad=\tau_3\Big(g_{a_3}F\big(t_{a_2}\big)\big(g_{\sigma_1}F(t_{a_2})\big)^{k}g_{\sigma_2}\big(F(t_{a_2})g_{\sigma_1}\big)^{h}\Big)\\ &\qquad=\tau_3\Big(g_{a_3}\big(F(t_{a_2})g_{\sigma_1}\big)^{k}F\big(t_{a_2}\big)g_{\sigma_2}\big(F(t_{a_2})g_{\sigma_1}\big)^{h}\Big),\end{aligned}$$

but, $g_{\sigma_2}=F\big(t_{a_2}^{-1}\big)g_{a_3}F\big(t_{a_2}\big)$, thus:

$$
\begin{aligned}
&\tau_3\bigg(g_{a_3}\Big(F(t_{a_2})g_{\sigma_1}\Big)^k g_{a_3}F\Big(t_{a_2}\Big)\Big(F(t_{a_2})g_{\sigma_1}\Big)^h\bigg)\\
&=\tau_3\bigg(g_{a_3}g^{-1}_{\sigma_2\sigma_1a_3}g_{\sigma_2\sigma_1a_3}\Big(F(t_{a_2})g_{\sigma_1}\Big)^k g_{a_3}F\Big(t_{a_2}\Big)\Big(F(t_{a_2})g_{\sigma_1}\Big)^h\bigg)\\
&=\tau_3\bigg(g^{-1}_{\sigma_2}\psi\Big[\Big(F(t_{a_2})g_{\sigma_1}\Big)^k\Big]g_{\sigma_2\sigma_1a_3}g_{a_3}F\Big(t_{a_2}\Big)\Big(F(t_{a_2})g_{\sigma_1}\Big)^h g^{-1}_{\sigma_1}\bigg),
\end{aligned}
$$

as we have done above, using the quadratic relations, we see that we are reduced to:

$$
\begin{aligned}
&\tau_3\bigg(g_{\sigma_2}\psi\Big[\Big(F(t_{a_2})g_{\sigma_1}\Big)^k\Big]g_{\sigma_2}g_{\sigma_1}F\Big(t_{a_2}\Big)\Big(F(t_{a_2})g_{\sigma_1}\Big)^h g^{-1}_{\sigma_1}\bigg)\\
&=\tau_3\bigg(g_{\sigma_2}\psi\Big[\Big(F(t_{a_2})g_{\sigma_1}\Big)^k\Big]g_{\sigma_2}g_{\sigma_1}F\Big(t^2_{a_2}\Big)\Big(g_{\sigma_1}F(t_{a_2})\Big)^{h-1}\bigg)\\
&=\Big(q-1\Big)\tau_3\bigg(g_{\sigma_2}\psi\Big[\Big(F(t_{a_2})g_{\sigma_1}\Big)^k\Big]g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^h\bigg)+\\
&\quad q\tau_3\bigg(g_{\sigma_2}\psi\Big[\Big(F(t_{a_2})g_{\sigma_1}\Big)^k\Big]g_{\sigma_2}g_{\sigma_1}\Big(g_{\sigma_1}F(t_{a_2})\Big)^{h-1}\bigg),
\end{aligned}
$$

the first term is covered by corollary 6.1.6. For the second we see that it is equal to:

$$
q\tau_3\bigg(g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^k g_{\sigma_2}g_{\sigma_1}\Big(g_{\sigma_1}F(t_{a_2})\Big)^{h-1}\bigg),\ \text{which is equal to}
$$

$$
\begin{aligned}
&q\Big(q-1\Big)\tau_3\bigg(g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^k g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^{h-1}\bigg)+\\
&q^2\tau_3\bigg(g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^k g_{\sigma_2}F\Big(t_{a_2}\Big)\Big(g_{\sigma_1}F(t_{a_2})\Big)^{h-2}\bigg),
\end{aligned}
$$

the first term is obviously, covered by corollary 6.1.6, for the second one we see that it is case 3 itself, but with $h-2$ instead of h. Thus, we get two elements for τ_3 to be evaluated on:

$$[\ a\ ]\ g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^k g_{\sigma_2}F\Big(t_{a_2}\Big)g_{\sigma_1}F\Big(t_{a_2}\Big),$$

$$[\ b\ ]\ g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^k g_{\sigma_2}F\Big(t_{a_2}\Big)\Big(g_{\sigma_1}F(t_{a_2})\Big)^2.$$

For [b] we can repeat what we have done until arriving to:

$$\tau_3\Bigg(g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^k g_{\sigma_2}g_{\sigma_1}g_{\sigma_1}F\Big(t_{a_2}\Big)\Bigg),\ \text{which is the following sum:}$$

$$\Big(q-1\Big)\tau_3\Bigg(\Big(g_{\sigma_1}F(t_{a_2})\Big)^k g_{\sigma_2}g_{\sigma_1}F\Big(t_{a_2}\Big)g_{\sigma_2}\Bigg)+q\tau_3\Bigg(\Big(g_{\sigma_1}F(t_{a_2})\Big)^k \underbrace{g_{\sigma_2}F\Big(t_{a_2}\Big)g_{\sigma_2}}_{F(t_{a_2})g_{\sigma_2}F(t_{a_2})}\Bigg),$$

obviously, the first term is covered by corollary 6.1.6, the second term is a Markov element.

For [a] we see that:

$$\begin{aligned}
&\tau_3\Bigg(g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^k g_{\sigma_2}F\Big(t_{a_2}\Big)g_{\sigma_1}F\Big(t_{a_2}\Big)\Bigg)\\
&=\tau_3\Bigg(g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^{k-1} g_{\sigma_1}\underbrace{F\Big(t_{a_2}\Big)g_{\sigma_2}F\Big(t_{a_2}\Big)}_{g_{\sigma_2}^2 g_{a_3}}g_{\sigma_1}F\Big(t_{a_2}\Big)\Bigg)\\
&=\ \Big(q-1\Big)\tau_3\Bigg(g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^{k-1} g_{\sigma_1}g_{\sigma_2}g_{a_3}g_{\sigma_1}F\Big(t_{a_2}\Big)\Bigg)+\\
&\quad q\tau_3\Bigg(g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^{k-1} g_{\sigma_1}g_{a_3}g_{\sigma_1}F\Big(t_{a_2}\Big)\Bigg),
\end{aligned}$$

the first term is covered by corollary 6.1.6, since it is equal to:

$$\Big(q-1\Big)\tau_3\Bigg(\Big(g_{\sigma_1}F(t_{a_2})\Big)^{k-1} g_{\sigma_1}F\Big(t_{a_2}\Big)g_{\sigma_2}g_{\sigma_1}F\Big(t_{a_2}\Big)g_{\sigma_2}\Bigg).$$

For the second term, we see that:

$$\begin{aligned}
&q\tau_3\Bigg(g_{\sigma_2}\Big(g_{\sigma_1}F(t_{a_2})\Big)^{k-1} g_{\sigma_1}g_{a_3}g_{\sigma_1}F\Big(t_{a_2}\Big)\Bigg)\\
&=q\tau_3\Bigg(\Big(g_{\sigma_1}F(t_{a_2})\Big)^{k-1} g_{\sigma_1}g_{a_3}g_{\sigma_1}F\Big(t_{a_2}\Big)g_{\sigma_2}\Bigg)
\end{aligned}$$

$$= q\tau_3\Big(\big(g_{\sigma_1}F(t_{a_2})\big)^{k-1}g_{\sigma_1}g_{a_3}g_{\sigma_1}g_{a_3}F\big(t_{a_2}\big)\Big)$$

$$= q\tau_3\Big(\big(g_{\sigma_1}F(t_{a_2})\big)^{k-1}g_{\sigma_1}^2g_{a_3}g_{\sigma_1}F\big(t_{a_2}\big)\Big),$$

which is a Markov element, since $g_{a_3} = F(t_{a_2})g_{\sigma_2}F(t_{a_2}^{-1})$.

4) We deal with $\tau_3\Big(g_{\sigma_2}\big(g_{\sigma_1}F(t_{a_2})\big)^kg_{\sigma_2}\big(F(t_{a_2})g_{\sigma_1}\big)^h\Big)$, using the same techniques:

$$\tau_3\Big(g_{\sigma_2}\big(g_{\sigma_1}F(t_{a_2})\big)^kg_{\sigma_2}\big(F(t_{a_2})g_{\sigma_1}\big)^h\Big)$$

$$= \tau_3\Big(g_{\sigma_2}\big(g_{\sigma_1}F(t_{a_2})\big)^kg_{\sigma_2}\big(F(t_{a_2})g_{\sigma_1}\big)^hg_{\sigma_2\sigma_1a_3}g_{\sigma_2\sigma_1a_3}^{-1}\Big)$$

$$= \tau_3\Big(g_{\sigma_2}\big(g_{\sigma_1}F(t_{a_2})\big)^kg_{\sigma_2}^2g_{\sigma_1}g_{a_3}\big(g_{\sigma_1}F(t_{a_2})\big)^hg_{\sigma_2\sigma_1a_3}^{-1}\Big),$$

so, we are reduced to:

$\tau_3\Big(\big(g_{\sigma_1}F(t_{a_2})\big)^{k-1}g_{\sigma_1}g_{a_3}\big(g_{\sigma_1}F(t_{a_2})\big)^hg_{a_3}F\big(t_{a_2}\big)\Big)$. Which is equal to:

$$\tau_3\Big(\big(g_{\sigma_1}F(t_{a_2})\big)^{k-1}\underbrace{g_{\sigma_1}g_{a_3}g_{\sigma_1}}_{V(g_{\sigma_1},g_{a_3})}F\big(t_{a_2}\big)\big(g_{\sigma_1}F(t_{a_2})\big)^{h-1}F\big(t_{a_2}\big)g_{\sigma_2}\Big),$$

for -1 and $-g_{\sigma_1}$ it is a Markov element. For $-g_{a_3}g_{\sigma_1}$ we see that:

$$\tau_3\Big(\big(g_{\sigma_1}F(t_{a_2})\big)^{k-1}g_{a_3}g_{\sigma_1}F\big(t_{a_2}\big)\big(g_{\sigma_1}F(t_{a_2})\big)^{h-1}F\big(t_{a_2}\big)g_{\sigma_2}\Big)$$

$$= \tau_3\Big(g_{a_3}F\big(t_{a_2}\big)\big(g_{\sigma_1}F(t_{a_2})\big)^{k-1}g_{a_3}\big(g_{\sigma_1}F(t_{a_2})\big)^h\Big),$$

which is covered by lemma 6.1.8.

For $-g_{a_3}$, we see that:

$$\tau_3\Big(\big(g_{\sigma_1}F(t_{a_2})\big)^{k-1}g_{a_3}F\big(t_{a_2}\big)\big(g_{\sigma_1}F(t_{a_2})\big)^{h-1}F\big(t_{a_2}\big)g_{\sigma_2}\Big)$$

$$=\tau_3\Big(\big(g_{\sigma_1}F(t_{a_2})\big)^{k-2}g_{\sigma_1}F\big(t_{a_2}^2\big)g_{\sigma_2}\big(g_{\sigma_1}F(t_{a_2})\big)^{h-2}g_{\sigma_1}F\big(t_{a_2}^2\big)g_{\sigma_2}\Big)$$

$$=\big(q-1\big)\tau_3\Big(\big(g_{\sigma_1}F(t_{a_2})\big)^{k-1}g_{\sigma_2}\big(g_{\sigma_1}F(t_{a_2})\big)^{h-2}g_{\sigma_1}F\big(t_{a_2}^2\big)g_{\sigma_2}\Big)+$$

$$q\tau_3\Big(\big(g_{\sigma_1}F(t_{a_2})\big)^{k-2}g_{\sigma_1}g_{\sigma_2}\big(g_{\sigma_1}F(t_{a_2})\big)^{h-2}g_{\sigma_1}F\big(t_{a_2}^2\big)g_{\sigma_2}\Big),$$

the first term is covered by corollary 6.1.6. We do the same thing with $F(t_{a_2}^2)$ in the second term, we arrive to:

$$q^2\tau_3\Big(\big(g_{\sigma_1}F(t_{a_2})\big)^{k-2}g_{\sigma_1}g_{\sigma_2}\big(g_{\sigma_1}F(t_{a_2})\big)^{h-2}g_{\sigma_1}g_{\sigma_2}\Big),$$

which is the case of lemma 6.1.7.

For $-g_{\sigma_1}g_{a_3}$ we see that:

$$\tau_3\Big(\big(g_{\sigma_1}F(t_{a_2})\big)^{k-1}g_{\sigma_1}g_{a_3}F\big(t_{a_2}\big)\big(g_{\sigma_1}F(t_{a_2})\big)^{h-1}F\big(t_{a_2}\big)g_{\sigma_2}\Big)$$

$$=\tau_3\Big(\big(g_{\sigma_1}F(t_{a_2})\big)^{k}g_{\sigma_2}\big(g_{\sigma_1}F(t_{a_2})\big)^{h-1}F\big(t_{a_2}\big)g_{\sigma_2}\Big)$$

$$=\big(q-1\big)\tau_3\Big(\big(g_{\sigma_1}F(t_{a_2})\big)^{k}g_{\sigma_2}\big(g_{\sigma_1}F(t_{a_2})\big)^{h-1}g_{\sigma_2}\Big)+$$

$$q\tau_3\Big(\big(g_{\sigma_1}F(t_{a_2})\big)^{k}g_{\sigma_2}\big(g_{\sigma_1}F(t_{a_2})\big)^{h-2}g_{\sigma_1}g_{\sigma_2}\Big),$$

corollary 6.1.6 covers the first term, while the second term is covered by (1) from our four cases.

Part 3

In this part we treat theorem 6.1.1 in the case where $n \geq 3$. By corollary 3.5.2, for $2 \leq n$, any element of the basis of $\widehat{TL}_{n+1}(q)$ is a linear combination of two kinds of elements, namely:

$$I = F_n(t_u)(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^k g_{\sigma_n\sigma_{n-1}..\sigma_s},$$

$$II = g_{\sigma_{i_0}..\sigma_2\sigma_1 a_{n+1}} F_n(t_u)(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^k g_{\sigma_n\sigma_{n-1}..\sigma_s},$$

here, u is in $W^c(\tilde{A_{n-1}})$, where $1 \leq s \leq n+1$ with $0 \leq i_0 \leq n-1$ and $0 \leq k$.

By lemma 6.1.3 we see that:

$$\begin{aligned}\Big(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}\Big)^k g_{\sigma_n} = (q-1)\Big(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}\Big)^k & \\ + \sum_{i=1}^{i=k-1} h_i\Big(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}\Big)^i & \\ + A\prod_{j=0}^{j=k-1}\phi^j\Big[\Big(\sigma_{n-1}\Big)^{-1}\Big] g_{\sigma_n}\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1} F(t_{a_n})\Big)^k &,\end{aligned}$$

but, $I = F_n(t_u)\underbrace{(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^k g_{\sigma_n}} g_{\sigma_{n-1}..\sigma_s}$, that is:

$$\begin{aligned}I = (q-1)F_n\Big(t_u\Big)\Big(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}\Big)^k g_{\sigma_{n-1}..\sigma_s} & \\ + \sum_{i=1}^{i=k-1} h_i F_n\Big(t_u\Big)\Big(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}\Big)^i g_{\sigma_{n-1}..\sigma_s} & \\ + A\prod_{j=0}^{j=k-1} F_n\Big(t_u\Big)\phi^j\Big[\Big(\sigma_{n-1}\Big)^{-1}\Big] g_{\sigma_n}\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1} F(t_{a_n})\Big)^k g_{\sigma_{n-1}..\sigma_s} &.\end{aligned}$$

Using the action of $g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}$ on $F_n(\widehat{TL}_n(q))$, we see that:

$$I = \sum_{i=1}^{i=k} F_n\Big(t_{b_i}\Big)\Big(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}\Big)^i + \sum_j F_n\Big(t_{b_j}\Big) g_{\sigma_n} F_n\Big(t_{d_j}\Big),$$

where b_j, c_j and d_i are in $W^c(\tilde{A_{n-1}})$, for every i and j.

Now, we see, as well, that:

$$II = g_{\sigma_{i_0}..\sigma_2\sigma_1} g_{a_{n+1}} F_n\Big(t_u\Big)\Big(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}\Big)^k g_{\sigma_n\sigma_{n-1}..\sigma_s}$$

$$= g_{\sigma_{i_0}..\sigma_2\sigma_1} F_n\Big(t_{a_n}\Big) \underbrace{g_{\sigma_n}\Big(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}\Big)^k} \psi^k\Big[F_n\Big(t_{a_n}^{-1}\Big)F_n\Big(t_u\Big)\Big] g_{\sigma_n\sigma_{n-1}..\sigma_s},$$

since $g_{a_{n+1}} = F_n(t_{a_n}) g_{\sigma_n} F_n(t_{a_n}^{-1})$.

By lemma 6.1.3, we see that II is equal to:

$$(q-1) g_{\sigma_{i_0}..\sigma_2\sigma_1} F_n\Big(t_{a_n}\Big)\Big(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}\Big)^k \psi^k\Big[F_n\Big(t_{a_n}^{-1}\Big)F_n(t_u)\Big] g_{\sigma_n\sigma_{n-1}..\sigma_s}$$

$$+ \sum_{i=1}^{i=k-1} g_{\sigma_{i_0}..\sigma_2\sigma_1} F_n\Big(t_{a_n}\Big) f_i \Big(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}\Big)^i \psi^k\Big[F_n(t_{a_n}^{-1})F_n(t_u)\Big] g_{\sigma_n\sigma_{n-1}..\sigma_s}$$

$$+ A g_{\sigma_{i_0}..\sigma_2\sigma_1} F_n\Big(t_{a_n}\Big)\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1} F(t_{a_n})\Big)^k g_{\sigma_n} \prod_{j=0}^{j=k-1} \psi^j\Big[F\Big((t_{a_n})^{-1}\Big)\Big]\psi^k\Big[F_n(t_{a_n}^{-1})F_n(t_u)\Big] g_{\sigma_n\sigma_{n-1}..\sigma_s},$$

which is equal to:

$$+ \sum_{i=1}^{i=k} F_n\Big(t_{x_i'}\Big)\Big(\underbrace{g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}\Big)^i g_{\sigma_n}} g_{\sigma_{n-1}..\sigma_s}$$

$$+ A g_{\sigma_{i_0}..\sigma_2\sigma_1} F_n\Big(t_{a_n}\Big)\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1} F(t_{a_n})\Big)^k g_{\sigma_n} \prod_{j=0}^{j=k-1} \psi^j\Big[F\Big((t_{a_n})^{-1}\Big)\Big]\psi^k\Big[F_n(t_{a_n}^{-1})F_n(t_u)\Big] g_{\sigma_n\sigma_{n-1}..\sigma_s},$$

where x_i' is in $W^c(\tilde{A_{n-1}})$ for all i.

Now we repeat the same step as for I, to get the next corollary.

Corollary 6.1.9. *Let $3 \leq n$. Let w be in $W^c(\tilde{A}_n)$.*

Then there exist $0 \leq k$ and $1 \leq s \leq n+1$. There exist x_i , y_i and z_i in $W^c(\tilde{A_{n-1}})$ $\Big[$With the convention $W^c(\tilde{A}_2) = W(\tilde{A}_2)\Big]$ such that:

$$g_w = \sum_{i=1}^{i=k} F_n\Big(t_{x_i}\Big)\Big(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}\Big)^i + \sum_j F_n\Big(t_{y_j}\Big) g_{\sigma_n} F_n\Big(t_{z_j}\Big) g_{\sigma_n\sigma_{n-1}..\sigma_s}.$$

Now we suppose that $3 \leq n$. Consider the following sequence:

$$\widehat{TL}_{n-1}(q) \longrightarrow \widehat{TL}_n(q) \longrightarrow \widehat{TL}_{n+1}(q).$$

We keep using g_{σ_i} (resp. t_{σ_i}) as generators of $\widehat{TL}_n(q)$ (resp. $\widehat{TL}_{n+1}(q)$). We use e_{σ_i} for $\widehat{TL}_{n-1}(q)$. With a simple computation, we see that g_{σ_n} commutes with $F_nF_{n-1}(e_{\sigma_i})$, for $1 \leq i \leq n-2$, and with $F_nF_{n-1}(e_{a_{n-1}})$, hence it commutes with every element in $F_nF_{n-1}(\widehat{TL}_{n-1}(q))$.

Lemma 6.1.4 and lemma 6.1.5 confirm that τ_{n+1} is uniquely defined over $\widehat{TL}_{n+1}(q)$ by its values on $g_{\sigma_n}(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n}))^k g_{\sigma_n}h$, for a positive k and an arbitrary h in $F(\widehat{TL}_n(q))$. In other terms: τ_{n+1} is uniquely defined over $\widehat{TL}_{n+1}(q)$ by its values over $g_{\sigma_n}\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n})\Big)^k g_{\sigma_n}F_n\Big(t_v\Big)$, with a positive k and an arbitrary v in $W^c(\tilde{A}_{n-1})$.

$$\text{Set } I := \tau_{n+1}\Bigg(g_{\sigma_n}\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n})\Big)^k g_{\sigma_n}F_n\Big(t_v\Big)\Bigg),$$

by corollary 6.1.9 we see that:

$$\begin{aligned} t_v = & \sum_{i=1}^{i=h} \underbrace{F_{n-1}\Big(e_{x_i}\Big)\Big(t_{\sigma_{n-1}\sigma_{n-2}..\sigma_1 a_n}\Big)^i}_{C} \\ & + \sum_j \underbrace{F_{n-1}\Big(e_{y_j}\Big)t_{\sigma_{n-1}}F_{n-1}\Big(e_{z_j}\Big)t_{\sigma_{n-1}\sigma_{n-2}..\sigma_s}}_{B} \\ & + \sum_j \underbrace{F_{n-1}\Big(e_{y'_j}\Big)t_{\sigma_{n-1}}F_{n-1}\Big(e_{z'_j}\Big)}_{A}, \end{aligned}$$

where $0 \leq h$ and $1 \leq s \leq n-1$. With x_i , y_i, z_i, y'_i and z'_i are in $W^c(\tilde{A}_{n-2})$.

Actually, we have added the third term C to the two terms of corollary 6.1.9, because we had to take into account here, the case of $s = n+1$, i.e., $g_{\sigma_{n+1}} = 1$ for $W^c(\tilde{A}_{n-1})$.

For terms of **Type (A)**, we see that:

$$\begin{aligned} I_1 &:= \tau_{n+1}\Bigg(g_{\sigma_n}\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F_n(t_{a_n})\Big)^k g_{\sigma_n}F_n\Big(F_{n-1}(e_{y'_j})t_{\sigma_{n-1}}F_{n-1}(e_{z'_j})\Big)\Bigg) \\ &= \tau_{n+1}\Bigg(\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F_n(t_{a_n})\Big)^k F_n\Big(F_{n-1}(e_{y'_j})\Big)g_{\sigma_n}F_n\Big(t_{\sigma_{n-1}}\Big)g_{\sigma_n}F_n\Big(F_{n-1}(e_{z'_j})\Big)\Bigg) \end{aligned}$$

$$= \tau_{n+1}\Big(\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n})\Big)^k F_n\Big(F_{n-1}(e_{y'_j})\Big)\underbrace{g_{\sigma_n}g_{\sigma_{n-1}}g_{\sigma_n}}_{r}F_n\Big(F_{n-1}(e_{z'_j})\Big)\Big)$$

$$= \tau_{n+1}\Big(\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n})\Big)^k F_n\Big(F_{n-1}(e_{y'_j})\Big)\underbrace{g_{\sigma_{n-1}}g_{\sigma_n}g_{\sigma_{n-1}}}_{r}F_n\Big(F_{n-1}(e_{z'_j})\Big)\Big),$$

which is clearly, the sum of values of τ_{n+1} on Markov elements, and elements in $F_n\Big(\widehat{TL}_n(q)\Big)$.

For terms of **Type (B)**, we see that:

$$I_2 := \tau_{n+1}\Big[g_{\sigma_n}\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n})\Big)^k g_{\sigma_n}F_n\Big(F_{n-1}(e_{y_j})t_{\sigma_{n-1}}F_{n-1}(e_{z_j})t_{\sigma_{n-1}\sigma_{n-2}..\sigma_s}\Big)\Big]$$

$$= \tau_{n+1}\Big[g_{\sigma_n}\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n})\Big)^k g_{\sigma_n}F_n\Big(F_{n-1}(e_{y_j})t_{\sigma_{n-1}}F_{n-1}(e_{z_j})t_{\sigma_{n-1}}t_{\sigma_{n-2}..\sigma_s}\Big)\Big]$$

$$= \tau_{n+1}\Big[g_{\sigma_n}F_nF_{n-1}\Big(e_{\sigma_{n-2}..\sigma_s}\Big)\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n})\Big)^k g_{\sigma_n}F_n\Big(F_{n-1}(e_{y_j})t_{\sigma_{n-1}}F_{n-1}(e_{z_j})t_{\sigma_{n-1}}\Big)\Big].$$

Now, we set ${}^m_rF := F_mF_{m-1}..F_r$.

We call δ the image of $F_{n-1}(e_{\sigma_{n-2}..\sigma_s})$ under the action of $\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n})\Big)^k$, thus:

$$I_2 = \tau_{n+1}\Big[g_{\sigma_n}\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n})\Big)^k g_{\sigma_n}F_n\Big(\delta\Big)\Big({}^n_{n-1}F(e_{y_j})\Big)t_{\sigma_{n-1}}\Big({}^n_{n-1}F(e_{z_j})\Big)g_{\sigma_{n-1}}\Big]$$

$$= \tau_{n+1}\Big[g_{\sigma_n}\Big(F_n(t_{\sigma_{n-1}\sigma_{n-2}..\sigma_1 a_n})\Big)^k g_{\sigma_n}F_n\Big(\delta\Big)\Big({}^n_{n-1}F(e_{y_j})\Big)t_{\sigma_{n-1}}\Big({}^n_{n-1}F(e_{z_j})\Big)g_{\sigma_{n-1}}\Big].$$

Now consider $(t_{\sigma_{n-1}\sigma_{n-2}..\sigma_1 a_n})^k$. We apply lemma 6.1.5 to this element in $\widehat{TL}_n(q)$, hence, it is the sum of two kind of elements: (1) Markov elements (2) elements of the form $t_{\sigma_{n-1}}(e_{\sigma_{n-2}\sigma_{n-3}..\sigma_1 a_{n-1}})^j t_{\sigma_{n-1}}\delta$, where $j \leq k$, and δ in $F_{n-1}\Big(\widehat{TL}_{n-1}(q)\Big)$. In the case (1) we are done. If we are in case (2), then we apply the lemma 6.1.5 on $(e_{\sigma_{n-2}\sigma_{n-3}..\sigma_1 a_{n-1}})^j$. We keep going in the same manner, by applying lemma 6.1.5 repeatedly (in fact $n-2$ times), we arrive to:

$$t_{\sigma_{n-1}}t_{\sigma_{n-2}}..t_{\sigma_2}\Big(F_{n-1}F_{n-2}..F_2({}^2g_{\sigma_1 a_2})^j\Big)t_{\sigma_2}..t_{\sigma_{n-2}}t_{\sigma_{n-1}}\lambda$$

$$= t_{\sigma_{n-1}}t_{\sigma_{n-2}}..t_{\sigma_2}\Big({}^{n-1}_2F({}^2g_{\sigma_1 a_2})^j\Big)t_{\sigma_2}..t_{\sigma_{n-2}}t_{\sigma_{n-1}}\lambda,$$

where λ is in ${}_{n-1}^{\;\;n}F(\widehat{TL}_{n-1}(q))$. We get:

$$I_2 = \tau_{n+1}\Big[g_{\sigma_n}F_n\Big(t_{\sigma_{n-1}}t_{\sigma_{n-2}}..t_{\sigma_2}\Big({}_2^{n-1}F\big({}^2g_{\sigma_1 a_2}\big)^j\Big)t_{\sigma_2}..t_{\sigma_{n-2}}t_{\sigma_{n-1}}\lambda\Big)g_{\sigma_n}F_n\Big(\delta\Big)$$
$$\Big({}_{n-1}^{n}F(e_{y_j})\Big)t_{\sigma_{n-1}}\Big({}_{n-1}^{n}F(e_{z_j})\Big)g_{\sigma_{n-1}}\Big]$$
$$= \tau_{n+1}\Big[g_{\sigma_n}g_{\sigma_{n-1}}g_{\sigma_{n-2}}..g_{\sigma_2}\Big({}_2^{n-1}F\big({}^2g_{\sigma_1 a_2}\big)^j\Big)g_{\sigma_2}..g_{\sigma_{n-2}}g_{\sigma_{n-1}}g_{\sigma_n}F_n(\lambda\delta)$$
$$\Big({}_{n-1}^{n}F(e_{y_j})\Big)t_{\sigma_{n-1}}\Big({}_{n-1}^{n}F(e_{z_j})\Big)g_{\sigma_{n-1}}\Big].$$

We set $M' := F_n\Big(\lambda\delta\Big)\Big({}_{n-1}^{n}F(e_{y_j})\Big)t_{\sigma_{n-1}}\Big({}_{n-1}^{n}F(e_{z_j})\Big)$, which is a Markov element in $\widehat{TL}_{n-1}(q)$. Hence, we have:

$$\tau_{n+1}\Big[g_{\sigma_n}g_{\sigma_{n-1}}g_{\sigma_{n-2}}..g_{\sigma_2}\Big({}_2^{n-1}F\big({}^2g_{\sigma_1 a_2}\big)^j\Big)g_{\sigma_2}..g_{\sigma_{n-2}}g_{\sigma_{n-1}}g_{\sigma_n}M'g_{\sigma_{n-1}}\Big]$$
$$= \tau_{n+1}\Big[\underbrace{g_{\sigma_{n-1}}g_{\sigma_n}g_{\sigma_{n-1}}}\,g_{\sigma_{n-2}}..g_{\sigma_2}\Big({}_2^{n-1}F\big({}^2g_{\sigma_1 a_2}\big)^j\Big)g_{\sigma_2}..g_{\sigma_{n-2}}g_{\sigma_{n-1}}g_{\sigma_n}M'\Big].$$

We apply the TL relations. The cases corresponding to 1 and $g_{\sigma_{n-1}}$ are obvious.

For the terms corresponding to $g_{\sigma_{n-1}}g_{\sigma_n}$, we have:

$$\tau_{n+1}\Big[g_{\sigma_{n-1}}g_{\sigma_n}\,\underbrace{g_{\sigma_{n-2}}..g_{\sigma_2}\Big({}_2^{n-1}F\big({}^2g_{\sigma_1 a_2}\big)^j\Big)g_{\sigma_2}..g_{\sigma_{n-2}}}\,g_{\sigma_{n-1}}g_{\sigma_n}M'\Big]$$
$$= \tau_{n+1}\Big[g_{\sigma_{n-1}}g_{\sigma_{n-2}}..g_{\sigma_2}\Big({}_2^{n-1}F\big({}^2g_{\sigma_1 a_2}\big)^j\Big)g_{\sigma_2}..g_{\sigma_{n-2}}\,\underbrace{g_{\sigma_n}g_{\sigma_{n-1}}g_{\sigma_n}}\,M'\Big].$$

We are done, since it is a sum of values of τ_{n+1} on Markov elements, and elements in $F_n\Big(\widehat{TL}_n(q)\Big)$. (the same for the term corresponding to g_{σ_n}) .

For the terms corresponding to $g_{\sigma_n}g_{\sigma_{n-1}}$, we have:

$$\tau_{n+1}\Big[g_{\sigma_n}g_{\sigma_{n-1}}g_{\sigma_{n-2}}..g_{\sigma_2}\Big({}_2^{n-1}F\big({}^2g_{\sigma_1 a_2}\big)^j\Big)g_{\sigma_2}..g_{\sigma_{n-2}}g_{\sigma_{n-1}}g_{\sigma_n}M'\Big],$$

which is the case of term (A), since M' is a Markov element.

For terms of **Type (C)**, we see that:

$$I_3 := \tau_{n+1}\Big[g_{\sigma_n}\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F_n(t_{a_n})\Big)^k g_{\sigma_n}F_n\Big(F_{n-1}(e_{x_i})(t_{\sigma_{n-1}\sigma_{n-2}..\sigma_1 a_n})^i\Big)\Big]$$

$$= \tau_{n+1}\Big[g_{\sigma_n}F_n\Big((t_{\sigma_{n-1}\sigma_{n-2}..\sigma_1 a_n})^k\Big)g_{\sigma_n}F_n\Big(F_{n-1}(e_{x_i})(t_{\sigma_{n-1}\sigma_{n-2}..\sigma_1 a_n})^i\Big)\Big].$$

Call γ the image of $F_{n-1}(e_{x_i})$ under the action of $(t_{\sigma_{n-1}\sigma_{n-2}..\sigma_1 a_n})^i$. Thus:

$$I_3 = \tau_{n+1}\Big[g_{\sigma_n}F_n\Big((t_{\sigma_{n-1}\sigma_{n-2}..\sigma_1 a_n})^k\Big)g_{\sigma_n}F_n\Big((t_{\sigma_{n-1}\sigma_{n-2}..\sigma_1 a_n})^i\gamma\Big)\Big].$$

As we have seen in the case (B), $(t_{\sigma_{n-1}\sigma_{n-2}..\sigma_1 a_n})^k$ can be written as some of elements of the form:

$$t_{\sigma_{n-1}}t_{\sigma_{n-2}}..t_{\sigma_2}\Big({}_2^{n-1}F({}^2g_{\sigma_1 a_2})^j\Big)t_{\sigma_2}..t_{\sigma_{n-2}}t_{\sigma_{n-1}}\lambda,$$

where $j \leq k$, and λ is in ${}_{n-1}^{\;n}F\Big(\widehat{TL}_{n-1}(q)\Big)$.

Call η the image of λ under the action of $(t_{\sigma_{n-1}\sigma_{n-2}..\sigma_1 a_n})^i$.

The determination of I_3 can be reduced to computing the following value:

$$\tau_{n+1}\Big[g_{\sigma_n}g_{\sigma_{n-1}}g_{\sigma_{n-2}}..g_{\sigma_2}\Big({}_2^{n-1}F({}^2g_{\sigma_1 a_2})^j\Big)g_{\sigma_2}..g_{\sigma_{n-2}}g_{\sigma_{n-1}}g_{\sigma_n}\Big(t_{\sigma_{n-1}\sigma_{n-2}..\sigma_1 a_n}\Big)^i\eta\gamma\Big].$$

We repeat the same Algorithm to $(t_{\sigma_{n-1}\sigma_{n-2}..\sigma_1 a_n})^i$. Hence, we get some $l \leq i$, and some Δ in ${}_{n-1}^{\;n}F\Big(\widehat{TL}_{n-1}(q)\Big)$, such that we are reduced to compute:

$$\tau_{n+1}\Big[g_{\sigma_n}g_{\sigma_{n-1}}g_{\sigma_{n-2}}..g_{\sigma_2}\Big({}_2^{n-1}F({}^2g_{\sigma_1 a_2})^j\Big)g_{\sigma_2}..g_{\sigma_{n-2}}\underbrace{g_{\sigma_{n-1}}g_{\sigma_n}g_{\sigma_{n-1}}}g_{\sigma_{n-2}}..g_{\sigma_2}$$

$$\Big({}_2^{n-1}F({}^2g_{\sigma_1 a_2})^l\Big)g_{\sigma_2}..g_{\sigma_{n-2}}g_{\sigma_{n-1}}\Delta\Big].$$

We see, after using the T-L relations, that the terms corresponding to 1 and $g_{\sigma_{n-1}}$ are values of τ_{n+1} on Markov elements.

The term corresponding to $g_{\sigma_{n-1}}g_{\sigma_n}$ is:

$$\tau_{n+1}\Big[g_{\sigma_n}g_{\sigma_{n-1}}g_{\sigma_{n-2}}..g_{\sigma_2}\Big({}_2^{n-1}F\big({}^2g_{\sigma_1 a_2}\big)^j\Big)g_{\sigma_2}..g_{\sigma_{n-2}}g_{\sigma_{n-1}}g_{\sigma_n}g_{\sigma_{n-2}}..g_{\sigma_2}$$

$$\Big({}_2^{n-1}F\big({}^2g_{\sigma_1 a_2}\big)^l\Big)g_{\sigma_2}..g_{\sigma_{n-2}}g_{\sigma_{n-1}}\Delta\Big]$$

$$=\tau_{n+1}\Big[g_{\sigma_{n-1}}g_{\sigma_{n-2}}..g_{\sigma_2}\Big({}_2^{n-1}F\big({}^2g_{\sigma_1 a_2}\big)^j\Big)g_{\sigma_2}..g_{\sigma_{n-2}}g_{\sigma_{n-1}}g_{\sigma_{n-2}}..g_{\sigma_2}$$

$$\Big({}_2^{n-1}F\big({}^2g_{\sigma_1 a_2}\big)^l\Big)g_{\sigma_2}..g_{\sigma_{n-2}}\underbrace{g_{\sigma_n}g_{\sigma_{n-1}}g_{\sigma_n}}\Delta\Big].$$

The term in square brackets is clearly a Markov element (the same thing with the term corresponding to g_{σ_n}) .

The term corresponding to $g_{\sigma_n}g_{\sigma_{n-1}}$ is:

$$\tau_{n+1}\Big[g_{\sigma_n}g_{\sigma_{n-1}}g_{\sigma_{n-2}}..g_{\sigma_2}\Big({}_2^{n-1}F\big({}^2g_{\sigma_1 a_2}\big)^j\Big)g_{\sigma_2}..g_{\sigma_{n-2}}g_{\sigma_n}g_{\sigma_{n-1}}g_{\sigma_{n-2}}..g_{\sigma_2}$$

$$\Big({}_2^{n-1}F\big({}^2g_{\sigma_1 a_2}\big)^l\Big)g_{\sigma_2}..g_{\sigma_{n-2}}g_{\sigma_{n-1}}\Delta\Big]$$

$$=\tau_{n+1}\Big[\underbrace{g_{\sigma_n}g_{\sigma_{n-1}}g_{\sigma_n}}g_{\sigma_{n-2}}..g_{\sigma_2}\Big({}_2^{n-1}F\big({}^2g_{\sigma_1 a_2}\big)^j\Big)g_{\sigma_2}..g_{\sigma_{n-2}}g_{\sigma_{n-1}}g_{\sigma_{n-2}}..g_{\sigma_2}$$

$$\Big({}_2^{n-1}F\big({}^2g_{\sigma_1 a_2}\big)^l\Big)g_{\sigma_2}..g_{\sigma_{n-2}}g_{\sigma_{n-1}}\Delta\Big].$$

It is a Markov element, theorem 6.1.1 follows.

6.1.2 $\widehat{TL}_2(q)$

In this subsection we parametrize all the traces on the algebra $\widehat{TL}_2(q)$, which have the same value over the two generators of $\widehat{TL}_2(q)$.

We have: $\widehat{TL}_2(q) = \widehat{H}_2(q)$ is generated by two elements: g_{σ_1}, g_{a_2}, with only the Hecke quadratic relations. That is:

$$g_{\sigma_1}^2 = (q-1)g_{\sigma_1} + q, \text{ thus: } g_{\sigma_1}^{-1} = \frac{1}{q}g_{\sigma_1} + \frac{1-q}{q}. \text{ The same for:}$$

$$g_{a_2}^2 = (q-1)g_{a_2} + q, \text{ thus: } g_{a_2}^{-1} = \frac{1}{q}g_{a_2} + \frac{1-q}{q}.$$

Making the same change as in 5.1.2, we set $\mathbf{T}_{\sigma_1} := \sqrt{q}g_{\sigma_1}$, and $\mathbf{T}_{a_2} := \sqrt{q}g_{a_2}$. Hence $\mathbf{T}_w = \left(\sqrt{q}\right)^{l(w)} g_w$ for any $w \in W(\tilde{A}_1)$. The set $\left\{\mathbf{T}_w; w \in W(\tilde{A}_1)\right\}$ is another K-basis of $H_n(q)$. The multiplication law of the new basis takes the form:

$$\mathbf{T}_{\sigma_1}^2 = \sqrt{q}(q-1)\mathbf{T}_{\sigma_1} + q^2 \text{ thus: } \mathbf{T}_{\sigma_1}^{-1} = \frac{1}{q^2}(\mathbf{T}_{\sigma_1} - \sqrt{q}(q-1)). \text{ The same for}$$

$$\mathbf{T}_{a_2}^2 = \sqrt{q}(q-1)\mathbf{T}_{a_2} + q^2 \text{ thus: } \mathbf{T}_{a_2}^{-1} = \frac{1}{q^2}(\mathbf{T}_{a_2} - \sqrt{q}(q-1))$$

As in 5.1.1, we set $f_{\sigma_1} := \frac{g_{\sigma_1}+1}{q+1}$ and $f_{a_2} := \frac{g_{a_2}+1}{q+1}$. In other terms $g_{\sigma_1} = (q+1)f_{\sigma_1} - 1$, and $g_{a_2} = (q+1)f_{a_2} - 1$. It is known that $\widehat{TL}_2(q)$ is generated by f_{σ_1} and f_{a_2} with relations $f_{\sigma_1}^2 = f_{\sigma_1}$ and $f_{a_2}^2 = f_{a_2}$. Moreover, $\widehat{TL}_2(q)$ has $\left\{f_w; w \in W(\tilde{A}_1)\right\}$ as a K-basis. The aim is to parametrize all traces over this algebra, which are invariant under the action of the Dynkin automorphism ψ_2, which exchanges $\mathbf{T}_{\sigma_1}$ and $\mathbf{T}_{a_2}$, (that is exchanging f_{σ_1} and f_{a_2}). Clearly, any trace has the same value on f_{σ_1} and f_{a_2} is invariant under the Dynkin automorphism ψ_2.

Proposition 6.1.10. *Let A_0, A_1 and α_i be arbitrary elements in the ground field for $1 \leq i$. Then, there exists a unique trace t on $\widehat{TL}_2(q)$, invariant by the action of ψ_2, such that: $A_0 = t(1)$, $A_1 = t(f_{\sigma_1})$ and $\alpha_s = t\Big((f_{\sigma_1 a_2})^s\Big)$.*

Proof. We start by the existence. Let t be the linear function given by:

$$\begin{aligned} t : \widehat{TL}_2(q) &\longrightarrow K \\ t(1) &= A_0 \\ t(f_{\sigma_1}) &= t(f_{a_2}) = A_1 \\ t\Big((f_{\sigma_1 a_2})^s\Big) &= t\Big((f_{a_2\sigma_1})^s\Big) = t\Big((f_{\sigma_1 a_2})^s f_{\sigma_1}\Big) = t\Big((f_{a_2\sigma_1})^s f_{a_2}\Big) = \alpha_s. \end{aligned}$$

Where A_0, A_1 and α_i are arbitrary elements in the ground field for $1 \leq i$.

We show that this linear function is a trace. First we see that t is, by definition, invariant under Dynkin automorphism (which is an involution in this case). In other terms: $t(z) = t\Big(\psi_2[z]\Big)$ for any z in $\widehat{TL}_2(q)$. In order to show that t is a trace, we show that $t(xy) = t(yx)$ for any x and y in $\widehat{TL}_2(q)$. The way to do so, is to show that it is true when x is any element of the left column, and y is any element of the right column, in the following table:

$[1](f_{\sigma_1 a_2})^k$	$[1'](f_{\sigma_1 a_2})^h$
$[2](f_{a_2\sigma_1})^k$	$[2'](f_{a_2\sigma_1})^h$
$[3](f_{\sigma_1 a_2})^k f_{\sigma_1}$	$[3'](f_{\sigma_1 a_2})^h f_{\sigma_1}$
$[4](f_{a_2\sigma_1})^k f_{a_2}$	$[4'](f_{a_2\sigma_1})^h f_{a_2}$

[1-1'], [2-2'], [3-3'] and [4-4']:

These are clear, since $k + h = h + k$ it follows directly.

[1-2']:

$$\text{Here, } t\Big(xy\Big) = t\Big((f_{\sigma_1 a_2})^k (f_{a_2\sigma_1})^h\Big) = t\Big((f_{\sigma_1 a_2})^k f_{a_2\sigma_1} (f_{a_2\sigma_1})^{h-1}\Big) = t\Big((f_{\sigma_1 a_2})^k (f_{\sigma_1 a_2})^{h-1} f_{\sigma_1}\Big)$$

$$= t\Big((f_{\sigma_1 a_2})^{k+h-1} f_{\sigma_1}\Big) = \alpha_{k+h-1},$$

$$\text{while, } t\Big(yx\Big) = t\Big((f_{a_2\sigma_1})^h (f_{\sigma_1 a_2})^k\Big) = t\Big((f_{a_2\sigma_1})^h f_{\sigma_1 a_2} (f_{\sigma_1 a_2})^{k-1}\Big) = t\Big((f_{a_2\sigma_1})^h (f_{a_2\sigma_1})^{k-1} f_{a_2}\Big)$$

$$= \alpha_{k+h-1}.$$

[1-3']:

$$\text{Here, } t(xy) = t\Big((f_{\sigma_1 a_2})^k (f_{\sigma_1 a_2})^h f_{\sigma_1}\Big) = t\Big((f_{\sigma_1 a_2})^{k+h} f_{\sigma_1}\Big), \text{ which is equal to } \alpha_{k+h},$$

$$\text{while, } t(yx) = t\Big((f_{\sigma_1 a_2})^h f_{\sigma_1} (f_{\sigma_1 a_2})^k\Big) = t\Big((f_{\sigma_1 a_2})^{h+k}\Big) = \alpha_{k+h}.$$

[1-4']:

Here, $t\Big(xy\Big) = t\Big((f_{\sigma_1 a_2})^k (f_{a_2\sigma_1})^h f_{a_2}\Big) = t\Big((f_{\sigma_1 a_2})^k f_{a_2} (f_{\sigma_1 a_2})^h\Big) = t\Big((f_{\sigma_1 a_2})^{k+h}\Big) = \alpha_{k+h}$,

while, $t\Big(yx\Big) = t\Big((f_{a_2\sigma_1})^h f_{a_2} (f_{\sigma_1 a_2})^k\Big) = t\Big(f_{a_2} (f_{\sigma_1 a_2})^h (f_{\sigma_1 a_2})^k\Big) = t\Big(f_{a_2} (f_{\sigma_1 a_2})^{h+k}\Big) = \alpha_{k+h}$.

[3-4']:

$$\text{We see that: } t\Big(xy\Big) = t\Big((f_{\sigma_1 a_2})^k f_{\sigma_1} (f_{a_2\sigma_1})^h f_{a_2}\Big) = t\Big((f_{\sigma_1 a_2})^{k+h+1}\Big) = \alpha_{k+h+1},$$

$$\text{with, } t\Big(yx\Big) = t\Big((f_{a_2\sigma_1})^h f_{a_2} (f_{\sigma_1 a_2})^k f_{\sigma_1}\Big) = t\Big((f_{a_2\sigma_1})^{h+k+1}\Big) = \alpha_{k+h+1}.$$

[3-1'], [3-2'], [4-1'] and [4-2']:

These case follows from [3-1'], [2-3'], [1-4'] and [2-4'] by exchanging h and k.

[4-3'], [2-1'], [2-3'] and [2-4']:

These cases follow from [3-4'], [1-2'], [1-4'] and [1-3'], resp, since t is invariant under the action of ψ_2.

Now, we end the proof by showing the uniqueness. Let t be a ψ_2-invariant trace on $\widehat{TL}_2(q)$. We have necessarily $t(f_{\sigma_1}) = t(f_{a_2})$, since t is a ψ_2-invariant, call this value A_1. For every $s \geq 1$ we have $t\Big((f_{\sigma_1 a_2})^s\Big) = t\Big((f_{a_2\sigma_1})^s\Big)$, since t is a trace, call this value α_s. Finally, we have $\alpha_s = t\Big((f_{\sigma_1 a_2})^s f_{\sigma_1}\Big) = t\Big((f_{a_2\sigma_1})^s f_{a_2}\Big)$, since t is a trace, and f_{a_2}, f_{σ_1} are idempotent. Call $t(1) = A_0$, thus, t is uniquely determined by A_0, A_1 and α_s, for $i \geq 1$. □

6.1.3 $\widehat{TL}_3(q)$

In this subsection, we parametrize all the traces over $\widehat{TL}_3(q)$, which are invariant under the action of the Dynkin automorphism ψ_3.

The affine Temperley-Lieb algebra in three generators $g_{\sigma_1}, g_{\sigma_2}$ and g_{a_3} can be presented by those generators with the relations of Hecke algebra, together with:

$$V(g_{\sigma_1}, g_{\sigma_2}) = V(g_{\sigma_1}, g_{a_3}) = V(g_{\sigma_2}, g_{a_3}) = 0.$$

We consider the same change of generators as in the case of $\widehat{TL}_2(q)$. Hence, $f_{\sigma_i} = \frac{g_{\sigma_i}+1}{q+1}$ with $g_{\sigma_i} = (q+1)f_{\sigma_i} - 1$ for $i = 1, 2$ the same for f_{a_3}. $\widehat{TL}_3(q)$ is presented by these three generators and the following relations:

$$f_{\sigma_i}^2 = f_{\sigma_i} \text{ for } i = 1, 2 \text{ and } f_{a_3}^2 = f_{a_3}.$$

$$f_{\sigma_i} f_{a_3} f_{\sigma_i} = \delta f_{\sigma_i} \text{ and } f_{a_3} f_{\sigma_i} f_{a_3} = \delta f_{a_3}.$$

$$f_{\sigma_1} f_{\sigma_2} f_{\sigma_1} = \delta f_{\sigma_1} \text{ and } f_{\sigma_2} f_{\sigma_1} f_{\sigma_2} = \delta f_{\sigma_2}.$$

Here, We will use the K-basis $\left\{f_w; w \in W^c(\tilde{A}_2)\right\}$.

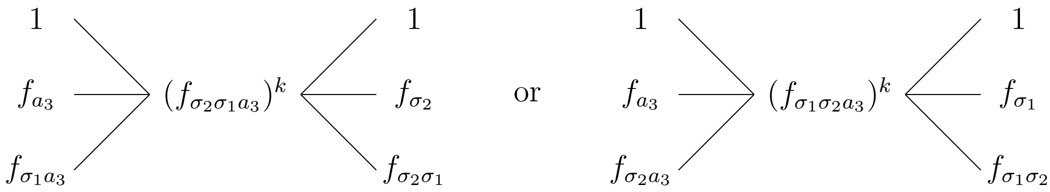
Lemma 6.1.11. *Let h and k be two positive integers. Then:*

$$\left(f_{\sigma_2\sigma_1 a_3}\right)^k \left(f_{\sigma_1\sigma_2 a_3}\right)^h = \begin{cases} \delta^{3h}\left(f_{\sigma_2\sigma_1 a_3}\right)^{k-h} & \textit{for } h < k. \\ \delta^{3k-1} f_{\sigma_2 a_3}\left(f_{\sigma_1\sigma_2 a_3}\right)^{k-h} & \textit{for } h \geq k. \end{cases}$$

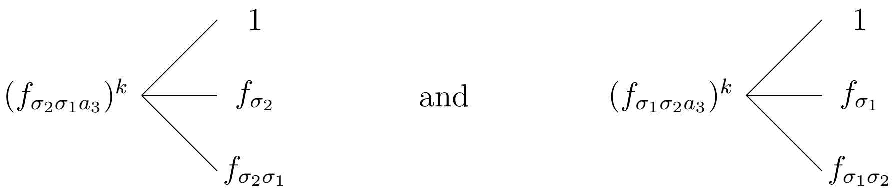
$$\left(f_{\sigma_1\sigma_2 a_3}\right)^h \left(f_{\sigma_2\sigma_1 a_3}\right)^k = \begin{cases} \delta^{3h}\left(f_{\sigma_1\sigma_2 a_3}\right)^{k-h} & \textit{for } h > k. \\ \delta^{3h-1} f_{\sigma_1 a_3}\left(f_{\sigma_2\sigma_1 a_3}\right)^{k-h} & \textit{for } h \leq k. \end{cases}$$

Proof. By induction, with a direct computation the lemma follows. □

Now we parametrize all the traces on $\widehat{TL}_3(q)$, which are invariant by the Dynkin automorphism ψ_3. We know that any element of the K-basis $\left\{f_w; w \in W^c(\tilde{A}_2)\right\}$ can be written as follows:



Lemma 6.1.12. *Let k be a positive integer, then for any w, such that $l(w) = 3k$, the element f_w is the image, under some power of the Dynkin automorphism ψ_3, of one of the following elements $(f_{\sigma_2\sigma_1 a_3})^k$ or $(f_{\sigma_1\sigma_2 a_3})^k$. The same for any u of length $3k+1$ (resp. $3k+2$), the element f_u is the image under a power of ψ_3 of one of the following elements $(f_{\sigma_2\sigma_1 a_3})^k f_{\sigma_2}$ or $(f_{\sigma_1\sigma_2 a_3})^k f_{\sigma_1}\Big($resp. $(f_{\sigma_2\sigma_1 a_3})^k f_{\sigma_2\sigma_1}$ or $(f_{\sigma_2\sigma_1 a_3})^k f_{\sigma_1\sigma_2}\Big)$.*



Proof. The proof is direct, by induction over k. □

Proposition 6.1.13. *For $1 \leq i$, let B_0, B_1, B_2 and β_i be in K. Then, there exists a unique, ψ_3-invariant, trace over $\widehat{TL}_3(q)$, say s, such that: $B_0 = s(1)$, $B_1 = s(f_{\sigma_1})$, $B_2 = s(f_{\sigma_1\sigma_2})$, $\beta_1 = s(f_{\sigma_1\sigma_2 a_3})$, $\beta_k = s((f_{\sigma_1\sigma_2 a_3})^k f_{\sigma_1})$ and $\beta_k = \frac{1}{\delta} s((f_{\sigma_1\sigma_2 a_3})^k f_{\sigma_1\sigma_2})$.*

Proof. For the existence, we consider the following linear map, we can show, using lemma 6.1.12, that it is indeed a ψ_3-invariant trace.

$$
\begin{array}{rcl}
s \text{ is given as follows, } s : \widehat{TL}_3(q) & \longrightarrow & K \\
s(1) & = & B_0, \\
s(f_{\sigma_1}) & = & s(f_{\sigma_2}) = s(f_{a_3}) = B_1, \\
s(f_u) & = & B_2 \text{ for any } u \text{ in } W^c(\tilde{A}_2) \text{ with } l(u) = 2,
\end{array}
$$

$$
\text{and } s(f_v) = \begin{cases} \beta_k & \text{when } \ l(v) = 3k \ \text{ or } \ l(v) = 3k+1. \\ \delta\beta_k & \text{when } \ l(v) = 3k+2. \end{cases}
$$

Where β_k (for $1 \leq k$), B_0, B_1 and B_2 are arbitrary in the field K. While for the uniqueness, we follow the steps of the proof of proposition 6.1.10

□

6.2 Affine Markov trace: existence and uniqueness

6.2.1 Towards a definition of affine Markov traces, existence

Let $1 \leq n$. Consider the following homomorphism:

$$\begin{aligned} F_n : \widehat{TL}_n(q) &\longrightarrow \widehat{TL}_{n+1}(q) \\ g_{\sigma_i} &\longmapsto g_{\sigma_i} \text{ for } 1 \leq i \leq n-1 \\ g_{a_n} &\longmapsto g_{\sigma_n} g_{a_{n+1}} g_{\sigma_n}^{-1}. \end{aligned}$$

In view of section 2.4.5, in particular proposition 2.4.15 we formulate the definition of an "affine" Markov trace.

Definition 6.2.1. *We call $(\hat{\tau}_n)_{1\leq i}$ an affine Markov trace, if every $\hat{\tau}_n$ is a trace function over $\widehat{TL}_n(q)$ with the following conditions*

- $\hat{\tau}_1(1) = 1$, *(here $\widehat{TL}_1(q) = K$).*
- $\hat{\tau}_{n+1}(F_n(h)T_{\sigma_n}^{\pm 1}) = \hat{\tau}_n(h)$. *for all $h \in \widehat{TL}_n(q)$ and for $n \geq 1$.*
- $\hat{\tau}_n$ *is invariant under the Dynkin automorphism ψ_n for all n.*

Remark 6.2.2. *We notice that the second condition gives us that $\hat{\tau}_{n+1}\Big(F_n(h)T_{\sigma_n}^{-1}\Big) = \hat{\tau}_n\Big(h\Big)$, which means that:*

$$\hat{\tau}_{n+1}\Big(F_n(h)[\frac{1}{q^2}T_{\sigma_n} - \frac{q-1}{q\sqrt{q}}]\Big) = \hat{\tau}_n\Big(h\Big). \text{ Thus } \hat{\tau}_{n+1}\Big(F_n(h)\Big) = -\frac{q+1}{\sqrt{q}}\hat{\tau}_n\Big(h\Big).$$

Remark 6.2.3. *The third condition of definition 6.2.1 is, in fact, not independent, i.e., it results from the first and second conditions. We just have to see that if we have two elements in $\widehat{TL}_n(q)$ (say x and y) such that $\psi_n(x) = y$, then $F_n(x)$ and $F_n(y)$ are conjugate in $\widehat{TL}_{n+1}(q)$, by some power of the element $g_{\sigma_n \dots \sigma_1 a_{n+1}}$ (we will explain that in the proof of the following proposition 6.2.4). Nevertheless, we will keep viewing it as a condition.*

Now, consider the following commutative diagram (see the end of subsection 5.2.3):

$$\begin{tikzcd}
\widehat{TL}_1(q) \arrow[r] \arrow[d, two heads] & \widehat{TL}_2(q) \arrow[r] \arrow[d, two heads] & \dots \arrow[r] & \widehat{TL}_n(q) \arrow[r] \arrow[d, two heads] & \widehat{TL}_{n+1}(q) \arrow[d, two heads] \\
TL_1(q) \arrow[r, hook] \arrow[drr, "\tau_1"'] & TL_2(q) \arrow[r, hook] \arrow[dr, "\tau_2"] & \dots \arrow[r, hook] & TL_{n-1}(q) \arrow[r, hook] \arrow[dl, "\tau_n"'] & TL_n(q) \arrow[dll, "\tau_{n+1}"] \\
 & & K & &
\end{tikzcd}$$

Set ρ_{n+1} to be the trace over $\widehat{TL}_{n+1}(q)$ induced by τ_{n+1} over $TL_n(q)$ for $0 \leq n$.

Proposition 6.2.4. *Under the above notations, we have:*

- $\rho_{n+1}(F_n(h)T_{\sigma_n}^{\pm 1}) = \rho_n(h)$. *for all* $h \in \widehat{TL}_n(q)$. *Where* $1 \leq n$.
- ρ_i *is invariant the action of* ϕ_i *for all* i.

$$\begin{tikzcd}
\widehat{TL_n(q)} \arrow[rrr, "F_n"] \arrow[dr, two heads, "e'_{n-1}"] \arrow[ddr, "\rho_n"'] & & & \widehat{TL_{n+1}(q)} \arrow[dl, two heads, "e'_n"'] \arrow[ddl, "\rho_{n+1}"] \\
 & TL_{n-1}(q) \arrow[r, hook, "x_n"] \arrow[d, "\tau_n"] & TL_n(q) \arrow[dl, "\tau_{n+1}"] & \\
 & K & &
\end{tikzcd}$$

Figure 6.1: T

Proof. We have: $\rho_{n+1}\Big(F_n(h)T_{\sigma_n}^{\pm 1}\Big)$ equals $\tau_{n+1}\left(e'_n\Big(F_n(h)\Big)e'_n\Big(T_{\sigma_n}^{\pm 1}\Big)\right)$.

$$\text{Hence, } \rho_{n+1}\Big(F_n(h)T_{\sigma_n}^{\pm 1}\Big) = \tau_{n+1}\left(x_n\Big(e'_{n-1}(h)\Big)T_{\sigma_n}^{\pm 1}\right) = \tau_n\Big(e'_{n-1}(h)\Big) = \rho_n\Big(h\Big).$$

We made use of the fact that the diagram T commutes, together with the fact that $(\tau_n)_{1\leq n}$ is a Markov trace.

For the second statement, we show that $\rho_n\Big(h\Big) = \rho_n\Big([h]\Big)$, where $[h]$ is the image of h under ϕ_n^{-1}. So we start from $\rho_n\Big(h\Big) = \tau_n\Big(e'_{n-1}(h)\Big)$. But since τ_n is the n-th Markov trace, we have $\tau_n\Big(e'_{n-1}(h)\Big) = -\frac{\sqrt{q}}{1+q}\tau_{n+1}\Big(x_n(e'_{n-1}(h))\Big)$, which is equal to $-\frac{\sqrt{q}}{1+q}\tau_{n+1}\left(e'_n\Big(F_n(h)\Big)\right)$, since the diagram T commutes, this term is equal to $-\frac{\sqrt{q}}{1+q}\rho_{n+1}\Big(F_n(h)\Big)$, hence to:

$$-\frac{\sqrt{q}}{1+q}\rho_{n+1}\Big(g_{\sigma_n..\sigma_1 a_{n+1}}F_n(h)g^{-1}_{\sigma_n..\sigma_1 a_{n+1}}\Big) = -\frac{\sqrt{q}}{1+q}\rho_{n+1}\Big(F_n([h])\Big).$$

Now we consider the same steps in the opposite direction, that is:

$$-\frac{\sqrt{q}}{1+q}\rho_{n+1}\Big(F_n([h])\Big) = -\frac{\sqrt{q}}{1+q}\tau_{n+1}\left(e'_n\Big(F_n([h])\Big)\right) = \rho_n\Big([h]\Big).$$

□

Corollary 6.2.5. *With the above notations, in the sense of definition 6.2.1: $(\rho_i)_{1\leq i}$ is an affine Markov trace over $\Big(\widehat{TL}_i(q)\Big)_{1\leq i}$.*

6.2.2 Uniqueness

Consider the following algebras homomorphism:

$$\begin{aligned} F_2 : \widehat{TL}_2(q) &\longrightarrow \widehat{TL}_3(q) \\ g_{\sigma_1} &\longmapsto g_{\sigma_1} \\ g_{a_2} &\longmapsto g_{\sigma_2}g_{a_3}g_{\sigma_2}^{-1}. \end{aligned}$$

We set $F := F_2$ in order to simplify in what follows. F can be expressed by the following form considering the "f" generators, we see that $F\Big(f_{a_2}\Big) = F\Big(\frac{g_{a_2}+1}{q+1}\Big)$, which is equal to $\frac{1}{q+1}g_{\sigma_2}g_{a_3}g_{\sigma_2}^{-1} + \frac{1}{q+1}$, hence to:

$$\frac{1}{q+1}\left[\Big((q+1)f_{\sigma_2}-1\Big)\Big((q+1)f_{a_3}-1\Big)\Big(\frac{1}{q}\Big((q+1)f_{\sigma_2}-1\Big)+\frac{1-q}{q}\Big)\right] + \frac{1}{q+1}.$$

Thus, we see that:

$$\begin{aligned} F : \widehat{TL}_2(q) &\longrightarrow \widehat{TL}_3(q) \\ f_1 &\longmapsto f_{\sigma_1} \\ f_{a_2} &\longmapsto -\frac{q+1}{q} f_{a_3\sigma_2} - (q+1) f_{\sigma_2 a_3} + f_{\sigma_2} + f_{a_3}. \end{aligned}$$

Notice that $F(f_{a_2}) f_{\sigma_2} F(f_{a_2}) = \delta F(f_{a_2})$, and $f_{\sigma_2} F(f_{a_2}) f_{\sigma_2} = \delta f_{\sigma_2}$. Since we are interested with viewing $F\Big(\widehat{TL}_2(q)\Big)$ in $\widehat{TL}_3(q)$, we will investigate in what follows, the elements $\Big(F(f_{\sigma_1} f_{a_2})\Big)^k$ and $\Big(F(f_{a_2} f_{\sigma_1})\Big)^k$, for k a positive integer.

Set $x_1 := F(f_{\sigma_1} f_{a_2}) = f_{\sigma_1} F(f_{a_2}) = -\dfrac{q+1}{q} f_{\sigma_1 a_3 \sigma_2} - (q+1) f_{\sigma_1 \sigma_2 a_3} + f_{\sigma_1\sigma_2} + f_{\sigma_1 a_3}$.

And for $1 \leq i$, we set:

$$\begin{aligned} x_i := \Big(-1\Big)^i \quad \Big(\frac{q+1}{q}\Big)^i \quad f^i_{\sigma_1 a_3 \sigma_2} \quad + \Big(-1\Big)^i \Big(q+1\Big)^i f^i_{\sigma_1\sigma_2 a_3} \\ + \Big(-1\Big)^{i-1} \Big(\frac{q+1}{q}\Big)^{i-1} f^{i-1}_{\sigma_1 a_3 \sigma_2} f_{\sigma_1 a_3} \quad + \Big(-1\Big)^{i-1} \Big(q+1\Big)^{i-1} f^{i-1}_{\sigma_1\sigma_2 a_3} f_{\sigma_1\sigma_2}. \end{aligned}$$

Notice that $x_1^2 = 3\delta x_1 + x_2$. It is easy to show that:

$$x_1 x_i = \delta^2 x_{i-1} + 2\delta x_i + x_{i+1}, \text{ for } 2 \leq i,$$

thus, for $1 \leq k$, we have $x_1^k = \sum\limits_{i=1}^{i=k-1} \gamma_i x_i + x_k$, here γ_i is a polynomial in δ, for all i.

Notice that $x_1 x_j = x_1 x_j$ for $j = 1, 2$. For $j = 1$ it is clear, while for $j = 2$ we have $x_2 = x_1^2 - 3\delta x_1$. Now suppose that $3 \leq j$. We have $x_j = x_1 x_{j-1} - \delta^2 x_{j-2} - 2\delta x_{j-1}$, hence we see by induction on j, that $x_1 x_j = x_1 x_j$, for all j .

We define the $\mathbb{Q}$-linear map $\chi : \widehat{TL}_3(q) \longrightarrow \widehat{TL}_3(q)$ which sends 1 to 1, and for any $u = s_1 s_2 .. s_r$ reduced expression of any element u in $W^c(\tilde{A}_2)$, it sends f_u to $f_{s_r s_{r-1} .. s_1}$, with q sent to $\frac{1}{q}$.

Set $z_1 := F(f_{a_2} f_{\sigma_1})$. Then

$$z_1 = F(f_{a_2}) f_{\sigma_1} = -\frac{q+1}{q} f_{a_3\sigma_2\sigma_1} - (q+1) f_{\sigma_2 a_3 \sigma_1} + f_{\sigma_2\sigma_1} + f_{a_3\sigma_1}.$$

And for $1 \leq i$, we set

$$\begin{aligned} z_i :&= \Big(-1\Big)^i \quad \Big(\frac{q+1}{q}\Big)^i \quad f^i_{a_3\sigma_2\sigma_1} &&+ \Big(-1\Big)^i\Big(q+1\Big)^i f^i_{\sigma_2a_3\sigma_1} \\ &+ \Big(-1\Big)^{i-1}\Big(\frac{q+1}{q}\Big)^{i-1} f_{\sigma_2\sigma_1}f^{i-1}_{a_3\sigma_2\sigma_1} &&+ \Big(-1\Big)^{i-1}\Big(q+1\Big)^{i-1} f_{a_3\sigma_1}f^{i-1}_{\sigma_2a_3\sigma_1}. \end{aligned}$$

Notice that $\chi(x_i) = z_i$ for all i. Now $\chi(x_1x_i) = \chi(x_ix_1) = \chi(z_1z_i)$. We see that $\chi(\delta) = \delta$. Moreover, $z_1z_j = \chi(x_1x_j) = \chi(\delta^2x_{i-1} + 2\delta x_i + x_{i+1}) = \delta^2z_{i-1} + 2\delta z_i + z_{i+1}$. And in the same way, by acting by χ, we find that $z_1^k = \sum\limits_{i=1}^{i=k-1} \gamma_i z_i + z_k$, where γ_i is as above.

Consider $x_if_{\sigma_2}$ for $1 \leq i$, we see that it is equal to:

$$\begin{aligned} &\Big(-1\Big)^i \quad \Big(\frac{q+1}{q}\Big)^i \quad f^i_{\sigma_1a_3\sigma_2} &&+ \Big(-1\Big)^i\Big(q+1\Big)^i f^i_{\sigma_1\sigma_2a_3}f_{\sigma_2} \\ &+ \Big(-1\Big)^{i-1}\Big(\frac{q+1}{q}\Big)^{i-1} f^{i-1}_{\sigma_1a_3\sigma_2}f_{\sigma_1a_3}f_{\sigma_2} &&+ \Big(-1\Big)^{i-1}\Big(q+1\Big)^{i-1} f^{i-1}_{\sigma_1\sigma_2a_3}f_{\sigma_1\sigma_2}, \end{aligned}$$

which is:
$$\begin{aligned} &\Big(-1\Big)^i \Big(\frac{q+1}{q}\Big)^i \quad f^i_{\sigma_1a_3\sigma_2} &&+ \delta\Big(-1\Big)^i\Big(q+1\Big)^i f^{i-1}_{\sigma_1\sigma_2a_3}f_{\sigma_1\sigma_2} \\ &+ \Big(-1\Big)^i\Big(\frac{q+1}{q}\Big)^{i-1} f^i_{\sigma_1a_3\sigma_2} &&+ \Big(-1\Big)^{i-1}\Big(q+1\Big)^{i-1} f^{i-1}_{\sigma_1\sigma_2a_3}f_{\sigma_1\sigma_2}. \end{aligned}$$

Hence, $x_if_{\sigma_2} = \Big[(-1)^{i-1}\dfrac{(q+1)^{i-1}}{q^i}\Big]f^i_{\sigma_1a_3\sigma_2} + \Big[(-1)^{i-1}(q+1)^{i-2}\Big]f^{i-1}_{\sigma_1\sigma_2a_3}f_{\sigma_1\sigma_2}$. For $1 \leq i$.

In particular $x_1f_{\sigma_2} = -\dfrac{q+1}{q}f_{\sigma_1a_3\sigma_2} - (q+1)f_{\sigma_1\sigma_2a_3}f_{\sigma_2} + f_{\sigma_1\sigma_2} + f_{\sigma_1a_3}f_{\sigma_2}$,

thus,

$$x_1f_{\sigma_2} = \frac{-1}{q}f_{\sigma_1a_3\sigma_2} + \frac{1}{q+1}f_{\sigma_1\sigma_2}.$$

Now we apply χ to $x_if_{\sigma_2}$. Hence

$$f_{\sigma_2}z_i = \Big[(-1)^iq(q+1)^{i-1}\Big]f^i_{\sigma_2a_3\sigma_1} + \Big[(-1)^{i-1}(\frac{q+1}{q})^{i-2}\Big]f_{\sigma_2\sigma_1}f^{i-1}_{a_3\sigma_2\sigma_1}. \text{ For } 1 \leq i.$$

In particular $f_{\sigma_2}z_1 = -qf_{\sigma_2a_3\sigma_1} + \frac{q}{q+1}f_{\sigma_2\sigma_1}$.

Take t to be any ψ_2-invariant trace over $\widehat{TL}_2(q)$, determined by A_0, A_1 and $(\alpha_i)_{1\leq i}$. Let s be anyψ_3-invariant trace over $\widehat{TL}_2(q)$, determined by B_0, B_1, B_2 and $(\beta_i)_{1\leq i}$. **We show in what follows that there are a unique t and a unique s, such that t is a second Markov trace in a given Markov trace, and s is the third Markov trace (associated to t).** So, in order to simplify, we set $\hat{\tau}_2 := t$ and $\hat{\tau}_3 := s$.

At first, being a first Markov trace, forces $\hat{\tau}_2$ to have the value 1 over $\mathbf{T}_{\sigma_1}$ and $\mathbf{T}_{a_2}$, but $f_{\sigma_1} = \frac{1+g_{\sigma_1}}{1+q} = \frac{1}{1+q} + \frac{\mathbf{T}_{\sigma_1}}{\sqrt{q}(1+q)}$. Hence, $A_1 = -\frac{\sqrt{q}}{1+q}$. Moreover, $\hat{\tau}_2(1) = -\frac{1+q}{\sqrt{q}}\hat{\tau}_1(1)$. Thus, $A_0 = -\frac{1+q}{\sqrt{q}}$.

Now, we have:

$$B_0 = \hat{\tau}_3\big(1\big) = -\frac{1+q}{\sqrt{q}}\hat{\tau}_2\big(1\big) = \big(-\frac{1+q}{\sqrt{q}}\big)^2,$$

$$\text{and } B_1 = \hat{\tau}_3\big(f_{\sigma_1}\big) = -\frac{1+q}{\sqrt{q}}\hat{\tau}_2\big(f_{\sigma_1}\big) = \frac{1+q}{\sqrt{q}}\frac{\sqrt{q}}{1+q} = 1.$$

Remark 6.2.6. *$\hat{\tau}_3$ must verify $\hat{\tau}_3\big(F(h)T_{\sigma_2}\big) = \hat{\tau}_2\big(h\big)$, for every h in $\widehat{TL}_2(q)$.*

$$\text{But, } \hat{\tau}_3\big(F(h)T_{\sigma_2}\big) = \sqrt{q}\hat{\tau}_3\big(F(h)g_{\sigma_2}\big) = \sqrt{q}\hat{\tau}_3\Big(F(h)\big[(q+1)f_{\sigma_2} - 1\big]\Big).$$

$$\begin{aligned}\text{So, } \sqrt{q}\hat{\tau}_3\Big(F(h)\big[(q+1)f_{\sigma_2} - 1\big]\Big) &= \sqrt{q}\big(q+1\big)\hat{\tau}_3\big(F(h)f_{\sigma_2}\big) - \sqrt{q}\hat{\tau}_3\big(F(h)\big)\\ &\sqrt{q}\big(q+1\big)\hat{\tau}_3\big(F(h)f_{\sigma_2}\big) + \sqrt{q}\frac{1+q}{\sqrt{q}}\hat{\tau}_2\big(h\big).\end{aligned}$$

Hence, our condition becomes

$$\sqrt{q}\big(q+1\big)\hat{\tau}_3\big(F(h)f_{\sigma_2}\big) = -\sqrt{q}\frac{1+q}{\sqrt{q}}\hat{\tau}_2\big(h\big) + \hat{\tau}_2\big(h\big) = -q\hat{\tau}_2\big(h\big).$$

Thus, we must have

$$\hat{\tau}_3\big(F(h)f_{\sigma_2}\big) = -\frac{\sqrt{q}}{(q+1)}\hat{\tau}_2\big(h\big), \text{ as an "f" equivalent to } \hat{\tau}_3\big(F(h)T_{\sigma_2}\big) = \hat{\tau}_2\big(h\big).$$

Now, we have:

$$B_2 = \hat{\tau}_3(f_{\sigma_1\sigma_2}) = -\frac{\sqrt{q}}{1+q}\hat{\tau}_2(f_{\sigma_1}) = (\frac{\sqrt{q}}{1+q})^2.$$

So, under the assumption that our two traces are the second and the third Markov traces in a given Markov trace, we get the following:

$$A_1 = -\frac{\sqrt{q}}{1+q},\ A_0 = -\tfrac{1+q}{\sqrt{q}}.$$

$$B_2 = \Big(\frac{\sqrt{q}}{1+q}\Big)^2,\ B_1 = 1 \text{ and } B_0 = \Big(\tfrac{1+q}{\sqrt{q}}\Big)^2.$$

In particular, we have for all $1 \leq i$:

$$\hat{\tau}_3\Big(x_1^i f_{\sigma_2}\Big) = -\frac{\sqrt{q}}{(q+1)}\hat{\tau}_2\Big((f_{\sigma_1 a_2})^i\Big), \quad \text{and} \quad \hat{\tau}_3\Big(z_1^i f_{\sigma_2}\Big) = -\frac{\sqrt{q}}{(q+1)}\hat{\tau}_2\Big((f_{a_2\sigma_1})^i\Big).$$

In other terms, for all i we have:

$$\hat{\tau}_3(x_1^i f_{\sigma_2}) = -\frac{\sqrt{q}}{(q+1)}\alpha_i, \quad \text{and} \quad \hat{\tau}_3(f_{\sigma_2} z_1^i) = -\frac{\sqrt{q}}{(q+1)}\alpha_i.$$

Since $\hat{\tau}_3$ is determined by β_i, we can view this equalities as system of equations in β_i and α_i. In what follows, we show that this system has at most one solution: $(\alpha_i, \beta_i)_{1\leq i}$.

For $i = 1$, we see that we have two equations:

$$\hat{\tau}_3(\frac{-1}{q}f_{\sigma_1 a_3\sigma_2} + \frac{1}{q+1}f_{\sigma_1\sigma_2}) = -\frac{\sqrt{q}}{(q+1)}\alpha_1, \quad \text{and} \quad \hat{\tau}_3(-qf_{\sigma_2 a_3\sigma_1} + \frac{q}{q+1}f_{\sigma_2\sigma_1}) = -\frac{\sqrt{q}}{(q+1)}\alpha_1,$$

$$\text{that is} \quad \frac{-1}{q}\beta_1 + \frac{1}{q+1}B_2 = -\frac{\sqrt{q}}{(q+1)}\alpha_1, \quad \text{and} \quad -q\beta_1 + \frac{q}{q+1}B_2 = -\frac{\sqrt{q}}{(q+1)}\alpha_1,$$

$$\text{that is} \quad \frac{-1}{q}\beta_1 + \frac{q}{(q+1)^3} = -\frac{\sqrt{q}}{(q+1)}\alpha_1, \quad \text{and} \quad -q\beta_1 + \frac{q^2}{(q+1)^3} = -\frac{\sqrt{q}}{(q+1)}\alpha_1.$$

Clearly, those two linear equations are independent, hence, they determine a unique solution (α_1, β_1). Let us see the equations when $i = 2$, we have:

$$\hat{\tau}_3(x_1^2 f_{\sigma_2}) = -\frac{\sqrt{q}}{(q+1)}\alpha_2, \quad \text{and} \quad \hat{\tau}_3(f_{\sigma_2} z_1^2) = -\frac{\sqrt{q}}{(q+1)}\alpha_2.$$

We see that:

$$x_1^2 f_{\sigma_2} = 3\delta x_1 f_{\sigma_2} + x_2 f_{\sigma_2} = 3\frac{-1}{q}\delta f_{\sigma_1 a_3 \sigma_2} + 3\frac{1}{q+1}\delta f_{\sigma_1\sigma_2} - \frac{(q+1)}{q^2} f^2_{\sigma_1 a_3 \sigma_2} - f_{\sigma_1\sigma_2 a_3} f_{\sigma_1\sigma_2}$$

$$= \frac{-3}{(1+q)^2} f_{\sigma_1 a_3 \sigma_2} + \frac{3}{(1+q)^3} f_{\sigma_1\sigma_2} - \frac{(q+1)}{q^2} f^2_{\sigma_1 a_3 \sigma_2} - f_{\sigma_1\sigma_2 a_3 \sigma_1\sigma_2},$$

$$\text{hence, } \hat{\tau}_3(F(x_1^2) f_{\sigma_2}) = \frac{-3}{(1+q)^2}\beta_1 + \frac{3}{(1+q)^3} B_2 - \frac{(q+1)}{q^2}\beta_2 - \delta\beta_1$$

$$= \frac{3}{(1+q)^3} B_2 - \frac{3+q}{(1+q)^2}\beta_1 - \frac{(q+1)}{q^2}\beta_2.$$

Now, $f_{\sigma_2} z_1^2 = \chi(x_1^2 f_{\sigma_2})$

$$= \chi\left(\frac{-3}{(1+q)^2}\right) f_{\sigma_2 a_3 \sigma_1} + \chi\left(\frac{3}{(1+q)^3}\right) f_{\sigma_2\sigma_1} - \chi\left(\frac{(q+1)}{q^2}\right) f^2_{\sigma_2 a_3 \sigma_1} - f_{\sigma_2\sigma_1 a_3 \sigma_2\sigma_1},$$

$$\text{so } f_{\sigma_2} z_1^2 = \frac{-3q^2}{(1+q)^2} f_{\sigma_2 a_3 \sigma_1} + \frac{3q^3}{(1+q)^3} f_{\sigma_2\sigma_1} - q(q+1) f^2_{\sigma_2 a_3 \sigma_1} - f_{\sigma_2\sigma_1 a_3 \sigma_2\sigma_1}.$$

$$\text{Now, we apply the trace } \hat{\tau}_3\Big(f_{\sigma_2} z_1^2\Big) = \frac{-3q^2}{(1+q)^2}\beta_1 + \frac{3q^3}{(1+q)^3} B_2 - q(q+1)\beta_2 - \delta\beta_1$$

$$= \frac{3q^3}{(1+q)^3} B_2 - \frac{3q^2+q}{(1+q)^2}\beta_1 - q(q+1)\beta_2.$$

In other terms, we have the two equations:

$$\begin{aligned} -\frac{(q+1)}{q^2}\beta_2 - \frac{3+q}{(1+q)^2}\beta_1 + \frac{3q}{(1+q)^5} &= -\frac{\sqrt{q}}{(q+1)}\alpha_2, \\ -q(q+1)\beta_2 - \frac{3q^2+q}{(1+q)^2}\beta_1 + \frac{3q^4}{(1+q)^5} &= -\frac{\sqrt{q}}{(q+1)}\alpha_2. \end{aligned}$$

Which indeed determine a unique (α_2, β_2) as a solution.

Now, have:

$$x_1^k = \sum_{i=1}^{i=k-1} \gamma_i x_i + x_k,$$

$$\text{hence, } x_1^k f_{\sigma_2} = \sum_{i=1}^{i=k-1} \gamma_i x_i f_{\sigma_2} + x_k f_{\sigma_2},$$

thus

$$x_1^k f_{\sigma_2} = \sum_{i=1}^{i=k-1} \gamma_i \left[\Big(-1 \Big)^{i-1} \frac{(q+1)^{i-1}}{q^i} \right] f^i_{\sigma_1 a_3 \sigma_2} + \gamma_i \left[\Big(-1 \Big)^{i-1} (q+1)^{i-2} \right] f^{i-1}_{\sigma_1 \sigma_2 a_3} f_{\sigma_1 \sigma_2}$$

$$+ \left[\Big(-1 \Big)^{k-1} \frac{(q+1)^{k-1}}{q^k} \right] f^k_{\sigma_1 a_3 \sigma_2} + \left[\Big(-1 \Big)^{k-1} (q+1)^{k-2} \right] f^{k-1}_{\sigma_1 \sigma_2 a_3} f_{\sigma_1 \sigma_2}.$$

Now we apply $\hat{\tau}_3$, we get:

$$-\frac{\sqrt{q}}{(q+1)} \alpha_k = \sum_{i=1}^{i=k-1} \gamma_i \left[\Big(-1 \Big)^{i-1} \frac{(q+1)^{i-1}}{q^i} \right] \beta_i + \delta \gamma_i \left[\Big(-1 \Big)^{i-1} (q+1)^{i-2} \right] \beta_{i-1}$$

$$+ \delta \left[\Big(-1 \Big)^{k-1} (q+1)^{k-2} \right] \beta_{k-1} + \left[\Big(-1 \Big)^{k-1} \frac{(q+1)^{k-1}}{q^k} \right] \beta_k.$$

It is clear that the coefficients of β_k is not zero, since β_k does not appear in:

$$A := \sum_{i=1}^{i=k-1} \gamma_i \left[\Big(-1 \Big)^{i-1} \frac{(q+1)^{i-1}}{q^i} \right] \beta_i + \delta \gamma_i \left[\Big(-1 \Big)^{i-1} (q+1)^{i-2} \right] \beta_{i-1}$$

$$+ \delta \left[\Big(-1 \Big)^{k-1} (q+1)^{k-2} \right] \beta_{k-1}.$$

Now, we repeat the same steps with z_i, namely:

$$z_1^k = \sum_{i=1}^{i=k-1} \gamma_i d_i + d_k,$$

$$\text{hence, } f_{\sigma 2} z_1^k = \sum_{i=1}^{i=k-1} \gamma_i f_{\sigma_2} d_i + f_{\sigma_2} d_k.$$

Thus,

$$f_{\sigma_2} z_1^k = \sum_{i=1}^{i=k-1} \gamma_i \Big[\Big(-1\Big)^i q(q+1)^{i-1}\Big] f^i_{\sigma_2 a_3 \sigma_1} + \gamma_i \Big[\Big(-1\Big)^{i-1} (\frac{q+1}{q})^{i-2}\Big] f_{\sigma_2\sigma_1} f^{i-1}_{a_3\sigma_2\sigma_1}$$

$$+\Big[\Big(-1\Big)^k q(q+1)^{k-1}\Big] f^k_{\sigma_2 a_3 \sigma_1} + \Big[\Big(-1\Big)^{k-1} (\frac{q+1}{q})^{k-2}\Big] f_{\sigma_2\sigma_1} f^{k-1}_{a_3\sigma_2\sigma_1}.$$

Now we apply $\hat{\tau}_3$, we get:

$$-\frac{\sqrt{q}}{(q+1)}\alpha_k = \sum_{i=1}^{i=k-1} \gamma_i \Big[\Big(-1\Big)^i q(q+1)^{i-1}\Big] \beta_i + \gamma_i \delta \Big[\Big(-1\Big)^{i-1} (\frac{q+1}{q})^{i-2}\Big] \beta_{i-1}$$

$$+\delta\Big[\Big(-1)^{k-1} (\frac{q+1}{q})^{k-2}\Big] \beta_{k-1} + \Big[\Big(-1\Big)^k q(q+1)^{k-1}\Big] \beta_k.$$

The coefficients of β_k is not zero, since β_k does not appear in

$$B := \sum_{i=1}^{i=k-1} \gamma_i \Big[\Big(-1\Big)^i q(q+1)^{i-1}\Big] \beta_i + \gamma_i \delta \Big[\Big(-1\Big)^{i-1} (\frac{q+1}{q})^{i-2}\Big] \beta_{i-1}$$

$$+\delta\Big[\Big(-1)^{k-1} (\frac{q+1}{q})^{k-2}\Big] \beta_{k-1}.$$

In other terms, we have the two following equations, in β_k and α_k:

$$-\frac{\sqrt{q}}{(q+1)}\alpha_k = A + \Big[\Big(-1\Big)^{k-1} \frac{(q+1)^{k-1}}{q^k}\Big] \beta_k,$$

$$-\frac{\sqrt{q}}{(q+1)}\alpha_k = B + \Big[\Big(-1\Big)^k q(q+1)^{k-1}\Big] \beta_k.$$

Those are two independent linear equations in β_k and α_k, with non-zero coefficients, by induction over k (that is: assuming that (α_i, β_i) is unique for $i < k$ then (α_k, β_k) is unique) we get the following corollary.

Corollary 6.2.7. * *Suppose that $(\hat{\tau}_i)_{1\leq i}$ is a Markov trace over the tower of $\tilde{A}$-type T-L algebras, then $\hat{\tau}_i = \rho_i$ for $i = 1, 2, 3$.*

Finally, we sum up the proof of the main theorem: we know, by corollary 6.2.5 that there exists, at least, an affine Markov trace. Now, corollary 6.2.7 says that in any given affine Markov trace, the three first traces are ρ_1, ρ_2 and ρ_3 (of corollary 6.2.5), while 6.1.1 affirms that a third trace in a given Markov trace determines a unique forth Markov trace, and so on for any $\hat{\tau}_i$ with $i \geq 3$. Hence, we get our main theorem:

Theorem 6.2.8. * *(Affine Markov trace)*

There exists a unique affine Markov trace on the tower of $\tilde{A}$-type Temperley-Lieb algebras, namely $(\rho_i)_{1 \leq i}$.

6.3 Bibliographical remarks and problems

In 2.4.1 we gave a new presentation of $B(\tilde{A}_n)$, this presentation gives a new presentation of $\widehat{H}_{n+1}(q)$, by generators and relations, with the very same relations, now, the question is: what will be the suitable relations, to define the associated presentation of $\widehat{TL}_{n+1}(q)$?, since we know that the images, of the "new generators", of $\widehat{H}_{n+1}(q)$ in $\widehat{TL}_{n+1}(q)$, indeed generate $\widehat{TL}_{n+1}(q)$, the answer would give a new point of view on $\widehat{TL}_{n+1}(q)$, even on the level of diagrammatic presentation.

It is clear that, we can improve theorem 6.1.1, by reducing the elements over which any trace is uniquely defined, for instance, it is clear that we can avoid elements fixed by ψ_{n+1} in $F_n(\widehat{TL}_n(q))$, as example. Now, the aim would be, to arrive to a "minimal" set of such elements (clearly it should be a proper subset of the set of Markov elements).

We call attention, to the fact that theorem 6.1.1 was more than "enough" in the proof of our main theorem 6.2.8, for we need just, the third part! actually we see that corollary 6.2.7 does not leave for ρ_3 a lot of choices, so we do not need to prove that it is unique, since we know that it is, already, so.

We point out the clear fact: the restrictions of the traces $(\rho_i)_{1\leq i}$ to the tower of A-type Temperley-Lieb algebras, is the trace of Ocneanu-Jones.

Our main question still stands: what does happened on the "Hecke" level?, i.e., "how much" there are of affine Markov traces on the tower of affine Hecke algebras? We know that there exists, already, one coming from our affine trace above, over the T-L tower. are there any others? If so, do they "all" come from the A-type Hecke algebra?, since we show that $\widehat{H}_{n+1}(q)$ surjects onto $H_n(q)$, we know that there are "z" distinguished affine trace! What about the B-Markov trace, do they come from them? For the moment we do not know, yet.

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